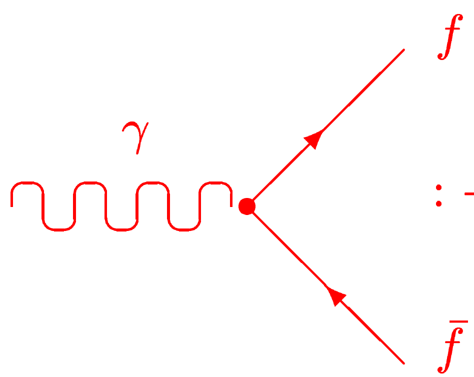
3.6 The Feynman-rules for QED

For any given action (Lagrangian) we can determine the Feynman-rules to write down the perturbative expansion of the Quantum Field Theory.

As Feynman-rules for QED, the Quantum Field Theory of electromagnatic interactions among charged fermions, one finds (see, e.g., Bjorken and Drell II):

- Draw all possible connected, topologically distinct Feynman-diagrams including loops with n external legs. Ignore vacuum-to-vacuum graphs.
- For each external photon with momentum k associate a polarization vector $\epsilon_\mu(k,\lambda)$, if it is ingoing, and $\epsilon^*_\mu(k,\lambda)$ for an outgoing photon.
- For each vertex of two fermions and a photon write



$: -ieQ_f\gamma_\mu$

where Q_f denotes the charge of the fermion (leptons: $Q_f = 1$).

- For each external fermion and anti-fermion draw a line with arrow, where the direction of the fermion and anti-fermion lines are opposite to each other. For each external (anti-)fermion with momentum p and spin s, write $u(p,s)(\bar{v}(p,s))$ for lines entering the graph and for lines leaving the graph write $\bar{u}(p,s)(v(p,s))$:

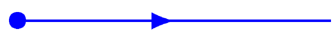
  e^- in initial state – ingoing electron line

  $u(p,s)$ 

  e^- in final state – outgoing electron line

  $\bar{u}(p,s)$ 

  e^+ in initial state – outgoing electron line

  $\bar{v}(p,s)$ 

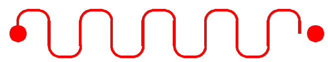
  e^+ in final state – ingoing electron line

  $v(p,s)$ 

- For each internal (virtual) fermion and anti-fermion with momentum k and mass m draw a line with arrow and associate a propagator:

 $$: \frac{i}{\gamma^\mu k_\mu - m + i\epsilon}$$

- For each internal (virtual) photon with momentum k associate a propagator, e.g., in Feynman-gauge:

 $: \dfrac{-i}{k^2 + i\epsilon} g_{\mu\nu}$

- The four-momentum is conserved at each vertex.
- For each internal momentum corresponding to a loop which is not fixed by momentum conservation at each vertex write $\int d^4k/(2\pi)^4$.
- For each closed fermion-loop multiply by (-1).
- A factor (-1) between two graphs which are only distinguished by interchanging two external identical fermion lines.

When writing down the expression for $\mathcal{M}_{fi}$ using Feynman-rules, start with an outgoing fermion line and continue going against the direction of the fermion line.

The last step: S-matrix elements $\rightarrow$ cross sections

The probability of the scattering of two particles with four-momenta q_1, q_2 and masses m_1, m_2 into n particles with four-momenta $p_1, p_2, \ldots p_n$ and masses $m_3, m_4, \ldots m_{n+2}$ is given by the differential cross section, $d\sigma$, as follows

$$d\sigma = \frac{1}{2\sqrt{[(2q_1 q_2)^2 - m_1^2 m_2^2]}} \prod_{i=1}^{n} [\frac{d^3 p_i}{(2\pi)^3 2p_i^0}]$$

$$(2\pi)^4 \delta^4(\sum_{i=1}^{n} p_i - q_1 - q_2) \overline{\sum} |\mathcal{M}_{fi}|^2$$

where $\overline{\sum}$ denotes averaging (summing) over the initial (final) state degrees of freedom (e.g., spin and color).

The invariant matrix element $\mathcal{M}_{fi}$ is connected to the interaction part of the S-matrix, $\mathcal{M}_{fi} =< p_1 \ldots p_n|T|q_1 \ldots q_m >$, and can be constructed using the Feynman-rules of the underlying Quantum Field Theory.

For a $2 \rightarrow 2$ scattering process of two particles with momenta q_1, q_2 and masses m_1, m_2 into 2 particles with momenta p_1, p_2 and masses m_3, m_4 , the differential cross section for the particle with momentum p_1 being scattered into the solid angle

$d\Omega = d\cos\theta d\phi$ reads

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{\sqrt{[(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2]}} \overline{\sum} |\mathcal{M}_{fi}|^2(s,t)$$

where we introduced the Lorentz-invariant kinematical variables (*Mandelstam variables*)

$$s = (q_1 + q_2)^2 = (p_1 + p_2)^2 \quad t = (q_1 - p_1)^2 = (q_2 - p_2)^2$$

$$u = (q_1 - p_2)^2 = (q_2 - p_1)^2$$

with $t = m_1^2 + m_3^2 - 2q_1^0 p_1^0 + 2|\vec{q}_1||\vec{p}_1|\cos\theta$.

It is convenient to work in the center-of-mass frame, where $\vec{q}_1 = -\vec{q}_2$, $\vec{p}_1 = -\vec{p}_2$ and s denotes the center-of-mass energy squared. In the limit of massless particles all momenta $|\vec{p}_i|$, $|\vec{q}_i|$ and energies p_i^0, q_i^0 are equal to $\sqrt{s}/2$. The total cross section is obtained by integrating over the solid angle

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \frac{d\sigma}{d\Omega}$$

An example: e^+e^- annihilation into 2 photons

There are two Feynman-diagrams contributing to this process and the corresponding matrix element reads:

$$\mathcal{M} = \bar{v}(q_2, s_2)(-ie\gamma_\mu)\frac{i}{(\not{q}_1 - \not{k}_1 - m + i\epsilon)}(-ie\gamma_\nu)u(q_1, s_1)$$

$$\epsilon^{\nu*}(k_1, \lambda_1)\epsilon^{\mu*}(k_2, \lambda_2) +$$

$$+\bar{v}(q_2, s_2)(-ie\gamma_\nu)\frac{i}{(\not{q}_1 - \not{k}_2 - m + i\epsilon)}(-ie\gamma_\mu)u(q_1, s_1)$$

$$\epsilon^{\nu*}(k_1, \lambda_1)\epsilon^{\mu*}(k_2, \lambda_2)$$

From this we obtain the spin averaged(summed) matrix element squared

$$\overline{\sum}|\mathcal{M}|^2 = e^4 \sum_{\lambda_1,\lambda_2} (\epsilon^{\nu*}(k_1, \lambda_1)\epsilon^{\sigma}(k_1, \lambda_1))(\epsilon^{\mu*}(k_2, \lambda_2)\epsilon^{\rho}(k_2, \lambda_2))$$

$$\frac{1}{4}\sum_{s_1,s_2}\left|\frac{\bar{v}\gamma_\mu(\not{q}_1 - \not{k}_1 + m)\gamma_\nu u}{(q_1 - k_1)^2 - m^2} + \frac{\bar{v}\gamma_\nu(\not{q}_1 - \not{k}_2 + m)\gamma_\mu u}{(q_1 - k_2)^2 - m^2}\right|^2 =$$

$$= e^4 g^{\mu\rho} g^{\nu\sigma}\frac{1}{4}\sum_{s_1,s_2}\left|\frac{\bar{v}\gamma_\mu(\not{q}_1 - \not{k}_1 + m)\gamma_\nu u}{(q_1 - k_1)^2 - m^2} + \frac{\bar{v}\gamma_\nu(\not{q}_1 - \not{k}_2 + m)\gamma_\mu u}{(q_1 - k_2)^2 - m^2}\right|^2$$

This is a general feature of calculating cross sections for processes involving external fermions, i.e. that one will encounter expressions of the form

$$\sum_{s_1,s_2} |\bar{v}(q_2,s_2)\Gamma^{\mu\nu}u(q_1,s_1)|^2$$

with, e.g., $\Gamma^{\mu\nu}=\gamma^\mu \not{k}\gamma^\nu$ or $\Gamma^{\mu\nu}=\gamma^\mu\gamma^\nu$. Writing the above expression in spinor components and using the spin sum for Dirac spinors one finds

$$\sum_{s_1,s_2} |\bar{v}(q_2,s_2)\Gamma^{\mu\nu}u(q_1,s_1)|^2 =$$

$$\sum_{s_1,s_2} [\bar{v}(q_2,s_2)\Gamma^{\mu\nu}u(q_1,s_1)][\bar{u}(q_1,s_1)\bar{\Gamma}^{\rho\sigma}v(q_2,s_2)] =$$

$$= \sum_{s_1,s_2} [v_d(q_2,s_2)\bar{v}_a(q_2,s_2)\Gamma^{\mu\nu}_{ab}u_b(q_1,s_1)\bar{u}_c(q_1,s_1)(\bar{\Gamma}^{\rho\sigma})_{cd}] =$$

$$= (\not{q}_2 - m_2)_{da}\Gamma^{\mu\nu}_{ab}(\not{q}_1 + m_1)_{bc}(\bar{\Gamma}^{\rho\sigma})_{cd} =$$

$$= Tr\{(\not{q}_2 - m_2)\Gamma^{\mu\nu}(\not{q}_1 + m_1)\bar{\Gamma}^{\rho\sigma}\}$$

For each continuous string of fermion lines in a Feynman-diagram such a trace over γ matrices has to be calculated.

The traces of Dirac matrices in 4 dimensions read:

$$Tr\{\gamma^\mu\} = 0 \text{ and for all other traces of odd numbers of } \gamma\text{'s}$$

$$Tr\{\gamma^\mu\gamma^\nu\} = 4g^{\mu\nu}$$

$$Tr\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}]$$

After having worked out the traces in our example we find:

$$\overline{\sum}|\mathcal{M}|^2 = 2e^4\left[\frac{(q_1k_2)}{(q_1k_1)} + \frac{(q_1k_1)}{(q_1k_2)} + 2m^2\left(\frac{1}{(q_1k_1)} + \frac{1}{(q_1k_2)}\right) + \right.$$
$$\left. -m^4\left(\frac{1}{(q_1k_1)} + \frac{1}{(q_1k_2)}\right)^2\right]$$

This yields the differential cross section in the center-of-mass-frame as follows ($\tau = 4m^2/s, \beta = \sqrt{1-\tau}$):

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\frac{1}{\beta}\left[\frac{1+\beta^2\cos^2\theta}{\tau+\beta^2\sin^2\theta} + \frac{2\tau}{\tau+\beta^2\sin^2\theta} - \frac{2\tau^2}{(\tau+\beta^2\sin^2\theta)^2}\right]$$

$$\rightarrow \frac{\alpha^2}{s}\left[\frac{1+\cos^2\theta}{\sin^2\theta}\right] \text{ for } \tau \rightarrow 0 \text{ and } \sin\theta \neq 0$$

Finally, the exciting part - the comparison with experiment:

see, e.g., M.Derrick et al., Phys.Rev.D34, 3286 (1986).