Lagrangians

Elementary Particle Physics

Strong Interaction Fenomenology

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Relativistic Notation

$$a^{\mu} = \left(a^0; a^1, a^2, a^3\right)$$

$$a_{\mu} = \left(a_0; a_1, a_2, a_3\right) = \left(a^0; -a^1, -a^2, -a^3\right)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$a_{\mu} = g_{\mu\nu} a^{\nu}$$

repeated indices are always summed

metric tensor

$$a_{\mu} b^{\mu} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

scalar product

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial t}; -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\partial^0; -\vec{\nabla}\right)$$

$$\partial_{\mu} a^{\mu} = \frac{\partial a^0}{\partial t} + \vec{\nabla} \cdot \vec{a} \qquad \partial^{\mu} \partial_{\mu} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \partial_0^2 - \nabla^2 = \partial_{\mu} \partial^{\mu}$$

Lagrangians

$\mathcal{L} = T - V$ Lagrangian

$S = \int_{t_1}^{t_2} \mathcal{L} dt$ Action. When S is minimized the Euler-Lagrange equations result and give rise to Newton's laws.

Example: Electrodynamics

$\vec{E}, \vec{B}$ Fields

$\vec{A}, V$ Potentials

$\vec{J}, \rho$ Currents

$$\boxed{\begin{aligned} \vec{E} &= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}}$$

$$A^{\mu} = \left(V;\vec{A}\right) = \left(A^0;\vec{A}\right)$$ Four-potential

$$J^{\mu} = \left(\rho;\vec{J}\right)$$ Four-current

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$ Electromagnetic field tensor

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i \qquad F^{ij} = \partial^i A^j - \partial^j A^i = \varepsilon^{ijk} B^k$$

Note that $F^{\mu\nu}$ is explicitly invariant under a transformation:

$$A^{\nu} \to A^{\nu} + \partial^{\nu}\chi$$

$$F^{\mu\nu} \to F^{\mu\nu} + \partial^{\mu}\partial^{\nu}\chi - \partial^{\nu}\partial^{\mu}\chi$$

$$\boxed{\mathcal{L} = \frac{1}{2}\left(E^2 - B^2\right) - \rho V + \vec{J}\cdot\vec{A}}$$

kinetic term

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_{\mu} A^{\mu}}$$

Interaction Lagrangian

Given this lagrangian the Euler-Lagrange equations become **Maxwell's equations**.

Lagrangians in Particle Physics

- Particle physics is formulated by giving the Lagrangian.
- The Lagrangian defines the theory.
  - It is written in terms of the elementary particles of the theory.
  - Any composite objects should appear as bound states that arise as solutions of the theory.
  - For electrodynamics the photon is the quantum of the electromagnetic field; it is represented by the vector potential field A^μ. The electron is represented by the fermion field ψ.
  - The interaction term is given by $J_\mu A^\mu$.
- More precisely: it is the potential energy parts of the Lagrangian that specify the theory.
- The Lagrangian is a single function that determines the dynamics and it must be a scalar in every relevant space, invariant under transformations, since the action is invariant.

The Real Scalar Field

$\phi(x)$ Real scalar field.

$$\mathcal{L} = \frac{1}{2}\left(\underbrace{\partial_\mu \phi \partial^\mu \phi}_{\text{kinetic term}} - \overbrace{m^2 \phi^2}^{\text{mass term}}\right)$$

This Lagrangian implies that ϕ satisfies a wave equation:

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \qquad \boxed{\text{Klein-Gordon equation}}$$

$$E^2 = \vec{p}^{\,2} + m^2 \qquad E = i\partial_0 \qquad \vec{p} = -i\vec{\nabla}$$

$$E^2 - \vec{p}^{\,2} = -\partial_0^2 + \vec{\nabla}^2 = -\partial_\mu \partial^\mu$$

$\mathcal{L}$ as defined so far is the Lagrangian density. The Lagrangian is given by:

$$L = \int \mathcal{L}(x,t) d^4 x$$

As an example let us construct a field normalized to a single quantum of definite energy ω and momentum $\vec{k}$.

$\phi = C\cos\left(\vec{k}\cdot\vec{r} - \omega t\right)$ solution of the wave equation if $\omega^2 = k^2 + m^2$

The Hamiltonian associated with a Lagrangian is: $H = \dot{\phi}\dfrac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L}$

Total energy: $E = \int H d^3x$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} \quad \Longrightarrow \quad H = \frac{1}{2}\left(\dot{\phi}^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2\right)$$

$$\Longrightarrow \quad E = \int\frac{1}{2}\left(\dot{\phi}^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2\right)d^3x$$

$$E = \frac{1}{2}C^2 \int \left[\left(\omega^2 + k^2\right)\sin^2\left(\vec{k}\cdot\vec{r} - \omega t\right) + m^2 \cos^2\left(\vec{k}\cdot\vec{r} - \omega t\right)\right] d^3x$$

$$= \frac{1}{2}C^2\left[\frac{1}{2}\left(\omega^2 + k^2\right) + \frac{1}{2}m^2\right]$$

$$= \frac{1}{2}C^2\omega^2$$

$$E = \hbar\omega \quad \Longrightarrow \quad C = \sqrt{\frac{2}{\omega}}$$

$$\phi = \frac{1}{\sqrt{2\omega}}\left[e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)} + e^{-i\left(\vec{k}\cdot\vec{r} - \omega t\right)}\right]$$

Creation and Destruction of Particles

If $|n_k\rangle$ is a state with n particles, all with energy ω and momentum $\vec{k}$ at time t :

$$|n_k\rangle \propto e^{in\omega t}$$

Then a state with one less particle will be described by: $|n_k - 1\rangle \propto e^{i(n-1)\omega t}$

$$\phi = \frac{1}{\sqrt{2\omega}}\left[a e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} + a^{\dagger} e^{-i\left(\vec{k}\cdot\vec{r}-\omega t\right)}\right]$$

$a^{\dagger}$ Creation operator

a Destruction operator

Every quantum field can create or destroy a particle. For a full derivation we need to go through the quatization of the scalar field, just as we go through the quantization of the electromagnetic field that leads to the interpretation of the photon as the quantum of the electromagnetic field.

Sources and Currents in non-Relativistic Quantum Theory

Schrödinger equation: $$2mi\frac{\partial\Psi}{\partial t}+\vec{\nabla}^2\Psi=0$$

Multiply this by $i\Psi^*$ and the complex conjugate by $-i\Psi$

Define: $$\rho=\left|\Psi\right|^2$$

$$\vec{J}=-\frac{i}{2m}\left(\Psi^*\vec{\nabla}\Psi-\Psi\vec{\nabla}\Psi^*\right)$$

we obtain: $$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{J}=0$$

For a free particle: $$\Psi=Ce^{i\left(\vec{p}\cdot\vec{r}-\omega t\right)}$$

$\boxed{\rho=C^2}$ Probability density

$\boxed{\vec{J}=\rho\frac{\vec{p}}{m}}$ Current density

Complex Scalar Field

Consider a system of two real scalar fields ϕ_1, ϕ_2 having the same mass m.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2\right) + \frac{1}{2}\left(\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2\right)$$

We can combine the two fields into a single complex scalar field ϕ:

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \qquad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$

The Lagrangian becomes:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

We “rotate” the two fields by and angle α:

$$\phi_1' = \phi_1 \cos\alpha + \phi_2 \sin\alpha$$
$$\phi_2' = -\phi_1 \sin\alpha + \phi_2 \cos\alpha$$

$$\phi' = \frac{\phi_1' + i\phi_2'}{\sqrt{2}}$$

$$\phi' = e^{-i\alpha}\phi$$

$$\phi'^* = e^{i\alpha}\phi^*$$

The Lagrangians does not change as a result of this transformation, since it only depends on $\phi^*\phi$. So the physics is invariant under this transformation.

Assume the rotation angle α is infinitesimal:

$$\phi' \approx (1 - i\alpha)\phi = \phi - i\alpha\phi \equiv \phi + \delta\phi$$

$$\delta\phi = -i\alpha\phi \qquad \delta\phi^* = i\alpha\phi^*$$

$$\boxed{\delta\mathcal{L} = \partial^\mu \underbrace{\left\{ \delta\phi \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} + (\phi \to \phi^*) \right\}}}$$

General result, not dependent on the details of the transformation

$\delta\mathcal{L} = 0 \Rightarrow$ conserved current

$$\delta\mathcal{L} = \alpha\partial^\mu S_\mu \qquad S_\mu = i\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right)$$

$$\delta\mathcal{L} = 0 \quad \Rightarrow \quad \partial^\mu S_\mu = 0$$

- $S_\mu \to -S_\mu$ if $\phi \to \phi^*$. In a relativistic theory if ϕ corresponds to a particle of electric charge e, ϕ^* corresponds to the antiparticle of electric charge – e and S_μ can be interpreted as a charge current density. We thus have a continuity equation expressing electric charge conservation.
- This description applies to any type of charge, not only electric.
- The above phase transformation is called a "global gauge transformation". If the parameter α could vary with the position and/or time $\alpha = \alpha(x,t)$ it would be a local gauge transformation.
- What we have seen is a specific example of a general property of quantum field theories: whenever a physical system is invariant under some transformation it leads to conserved quantities.

$$Q(t) = \int S_0(x) d^3x \qquad \frac{dQ}{dt} = 0$$

- A physical realization of such a system is given by neutral kaons. K_1 and K_2 are like ϕ_1 and ϕ_2; K^0 and $\bar{K}^0$ are like ϕ and ϕ^*. The charge is the strangeness. For the kaons "charge" conservation is broken by weak interactions which can convert $K \leftrightarrow \bar{K}$ and consequently introduces a small level splitting, i.e. the mass difference between the K_1 and K_2.
- In quantum theory higher order radiative corrections can give a result different from zero for the divergence of S_μ. Such terms are called anomalies.

Interactions

Let us add to the Lagrangian an interaction term, such as:

$$\mathcal{L}_{\text{int}} = -\phi\rho(\vec{x}, t)$$

The Klein-Gordon equation acquires a source term:

$$\partial_\mu \partial^\mu \phi + m^2 \phi = \rho$$

To study how the system behaves we examine the simple case:

$$\rho = g\delta(\vec{x})$$

The problem can be 'solved' by a Fourier transform procedure.

$$\left(-\vec{\nabla}^2 + m^2\right)\phi = g\delta(\vec{x})$$

$$\phi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}(\vec{k}) \qquad \tilde{\phi}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \phi(\vec{x})$$

$$\phi(\vec{x}) = \frac{g}{(2\pi)^3} \int d^3k \, \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 + m^2}$$

$$\boxed{\phi = \frac{g}{4\pi} \frac{e^{-mr}}{r}}$$

Yukawa identified ϕ as a meson field, with the nucleon as source. In analogy to the electromagnetic field, transmitted by photons, this meson field should be transmitted by particles (mesons). When the particles have a mass m the field has a range r

$$r \sim 1/m$$

Interaction Hamiltonian between two nucleons,the second described by $\rho_2(\vec{x})$

$$H = -\int d^3x \phi(\vec{x}) \rho_2(\vec{x})$$

$$\phi(\vec{x}) = \frac{1}{4\pi} \int d^3x' \rho_1(\vec{x}') \frac{e^{-m|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \qquad \rho_1(\vec{x}') = g\delta(\vec{x}')$$

$$H_{12} = \frac{1}{4\pi} \int d^3x d^3x' \rho_1(\vec{x}) \rho_2(\vec{x}') \frac{e^{-m|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

Hence the potential can be written:

$$V(r) = -\frac{1}{4\pi} \frac{e^{-mr}}{r}$$

In a quantum field theory all interactions are due to the exchange of quanta. In momentum space the quantity representing the exchanged particle of mass m is the propagator:

$$\boxed{\frac{1}{k^2 - m^2}}$$

Summary of the Lagrangians

Real spin-zero field of mass m (scalar or pseudoscalar)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \qquad \partial_\mu \partial^\mu \phi + m^2 \phi^2 = 0$$

Complex (pseudo)scalar field of mass m
(or two real scalars of the same mass) ϕ_1, ϕ_2

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2\right) + \frac{1}{2}\left(\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2\right)$$

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \qquad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$

$$\mathcal{L} = \left(\partial_\mu \phi\right)^* \partial^\mu \phi - m^2 \phi^* \phi$$

spin-½ fermion field of mass m

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0$$ Dirac equation

Abelian vector field of mass m

the following mass term is added to the Lagrangian:

$$\frac{1}{2}m^2 B^{\mu} B_{\mu}$$

Non-Abelian vector field

Examples of non-abelian vector fields are gluons and the W boson. The vector potential is generalized to a non-abelian field by adding an internal index a: W_a^μ

For SU(2) a = 1,2,3 whereas for SU(3) a = 1,2, ... , 8.

We then define (using W as an example):

$$W_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu + g f_{abc} W_b^\mu W_c^\nu$$

The f_{abc} are structure constants. This form of the field tensor is invariant under gauge transformations. The Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} W_a^{\mu\nu} W_{\mu\nu}^a + \frac{1}{2} m^2 W_a^\mu W_\mu^a$$