Mitglied der Helmholtz-Gemeinschaft

JÜLICH FORSCHUNGSZENTRUM

Lecture on Anomalies and the Infinite Hotel

Ferrara International School Niccolò Cabeo 2015: *Infinities*

Ferrara | May 25, 2015 | Andreas Wirzba

Infinite Hotel?

Infinite Hotel?

www.hotelcarlton.net/fotogallery-carlton

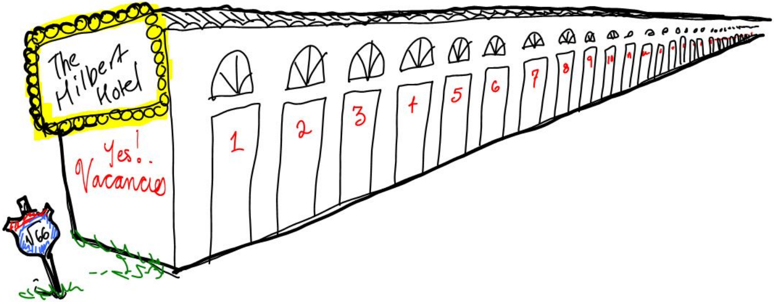
Infinite Hotel?

www.hotelcarlton.net/fotogallery-carlton

Not really

Infinite Hotel?

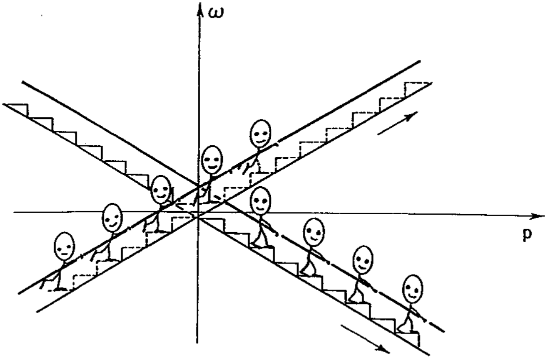
Rather in the spirit of David Hilbert's lecture from 1924



komplexify.com/blog/2014/05/14

Infinite Hotel?

or in the spirit of the *infinite escallators*



H.B. Nielsen & A.W., NBI-HE-87-32 (1987)

What is an anomaly?

In short: *A quantum-mechanical obstruction to a classical conservation law*

What is an anomaly?

In short: *A quantum-mechanical obstruction to a classical conservation law*

Outline of the lecture:

I. Anomalies $\boxed{\checkmark}$

(a) Warm-up: index of a harmonic oscillator ✓
(b) The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story ✓
(c) Adler-Bell-Jackiw (chiral) anomaly in 3+1 D ✓
(d) Digressions: the scale anomaly etc.✓
(e) Application: the $\pi^0 \to \gamma\gamma$ decay ✓

II. From the $U(1)_A$ problem to axions $\boxed{}$

(a) $U(1)_A$ problem and instantons✓
(b) θ vacuum and strong *CP* problem ✓
(c) Peccei-Quinn mechanism and axions $\boxed{}$
(d) Invisible axions $\boxed{}$
(e) The Empire strikes back (!?) ✓

PART I

Warm-up: index relation for single harmonic oscillator

What is **wrong** with the following calculation?:

$$\mathrm{Tr}\left([a, a^\dagger]\right) = \mathrm{Tr}\left(aa^\dagger - a^\dagger a\right) \stackrel{?}{=} \mathrm{Tr}\left(aa^\dagger - aa^\dagger\right) = \mathrm{Tr}\left(aa^\dagger\right) - \mathrm{Tr}\left(aa^\dagger\right) = 0\ ?$$

Warm-up: index relation for single harmonic oscillator

What is **wrong** with the following calculation?:

$$\mathrm{Tr}\left([a, a^\dagger]\right) = \mathrm{Tr}\left(aa^\dagger - a^\dagger a\right) \stackrel{?}{=} \mathrm{Tr}\left(aa^\dagger - aa^\dagger\right) = \mathrm{Tr}\left(aa^\dagger\right) - \mathrm{Tr}\left(aa^\dagger\right) = 0\,?$$

Only true if the a and $a^\dagger$ Hilbert spaces are **truncated** at the **same** order (*i.e.* *only* matrix elements $\langle n{+}1|a^\dagger|n\rangle$ and $\langle n|a|n{+}1\rangle$ with $n = 0, 1, 2, \ldots, N_{\mathrm{trunc}}$):

$$\mathrm{Tr}_{\mathrm{trunc}}\left(a_{\mathrm{t}}\, a_{\mathrm{t}}^\dagger - a_{\mathrm{t}}^\dagger a_{\mathrm{t}}\right) = \sum_{n=0}^{N_{\mathrm{trunc}}} (n{+}1) - \sum_{n=1\,(\to 0)}^{N_{\mathrm{trunc}}+1} n \;=\; \sum_{n=0}^{N_{\mathrm{trunc}}} (n{+}1) - \sum_{m=0}^{N_{\mathrm{trunc}}} (m{+}1) = 0\,.$$

Warm-up: index relation for single harmonic oscillator

What is **wrong** with the following calculation?:

$$\mathrm{Tr}\left([a,a^\dagger]\right) = \mathrm{Tr}\left(aa^\dagger - a^\dagger a\right) \stackrel{?}{=} \mathrm{Tr}\left(aa^\dagger - aa^\dagger\right) = \mathrm{Tr}\left(aa^\dagger\right) - \mathrm{Tr}\left(aa^\dagger\right) = 0\,?$$

Only true if the a and $a^\dagger$ Hilbert spaces are **truncated** at the **same** order (*i.e. only* matrix elements $\langle n{+}1|a^\dagger|n\rangle$ and $\langle n|a|n{+}1\rangle$ with $n = 0, 1, 2, \ldots, N_{\text{trunc}}$):

$$\mathrm{Tr}_{\text{trunc}}\left(a_t a_t^\dagger - a_t^\dagger a_t\right) = \sum_{n=0}^{N_{\text{trunc}}} (n{+}1) - \sum_{n=1(\to 0)}^{N_{\text{trunc}}+1} n = \sum_{n=0}^{N_{\text{trunc}}} (n{+}1) - \sum_{m=0}^{N_{\text{trunc}}} (m{+}1) = 0\,.$$

In the infinite limit of the truncation parameter a suitable regularization needed:

$$\mathrm{Tr}\left(\left[a,a^\dagger\right]\left(e^{-s\,a^\dagger a} - e^{-s\,aa^\dagger}\right)\right) = \mathrm{Tr}\left(e^{-s\,a^\dagger a} - e^{-s\,aa^\dagger}\right)$$

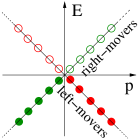
$$= \mathrm{Tr}\left(e^{-s\,a^\dagger a}\right) - \mathrm{Tr}\left(e^{-s\,aa^\dagger}\right) = \sum_{n=0}^{\infty} e^{-sn} - \sum_{n=0}^{\infty} e^{-s(n+1)} = 1 \quad \forall s > 0$$

$$= \dim \mathrm{Ker}(a) - \dim \mathrm{Ker}(a^\dagger) \equiv \mathrm{Index}(a) \quad \text{since}\ \ \mathrm{Ker}(a) = \{|0\rangle\}\ \text{but}\ \ \mathrm{Ker}(a^\dagger) = \varnothing.$$

Vanishing index in truncated space, but recovery of *non-zero index* in infinite (hotel) limit: one of the characteristic properties of the quantum anomaly.

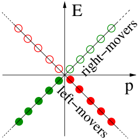
The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:



The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

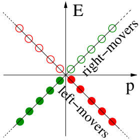
1. Dispersion for massless fermions in 1+1 D:



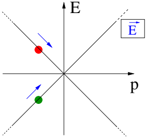
2. Add electric field
   to a single right/left-mover:

The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

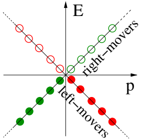


2. Add electric field
   to a single right/left-mover:

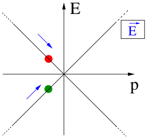


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

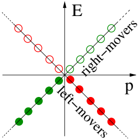


2. Add electric field
   to a single right/left-mover:

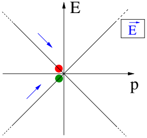


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

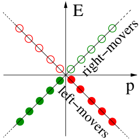


2. Add electric field
   to a single right/left-mover:

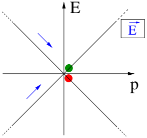


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

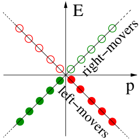


2. Add electric field
   to a single right/left-mover:

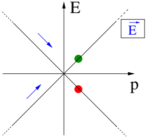


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

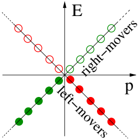


2. Add electric field
   to a single right/left-mover:

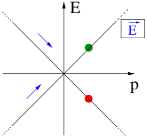


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:



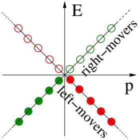
2. Add electric field
to a single right/left-mover:



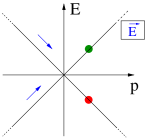
Nothing happens to single states, except energy change.

The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

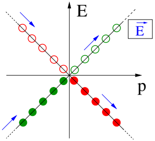


2. Add electric field
   to a single right/left-mover:



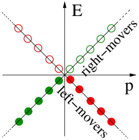
Nothing happens to single states, except energy change.

3. However, for *infinite mode sums* the level shifts induce a surplace of right-movers:

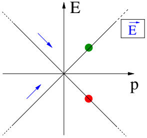


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

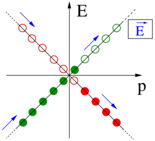


2. Add electric field to a single right/left-mover:



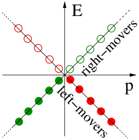
Nothing happens to single states, except energy change.

3. However, for *infinite mode sums* the level shifts induce a surplace of right-movers:

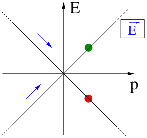


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

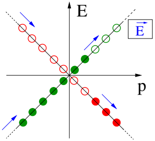


2. Add electric field
   to a single right/left-mover:



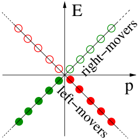
Nothing happens to single states, except energy change.

3. However, for *infinite mode sums* the level shifts induce a surplace of right-movers:

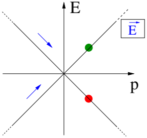


The $U(1)_A$ anomaly in 1+1 D – an infinite hotel story

1. Dispersion for massless fermions in 1+1 D:

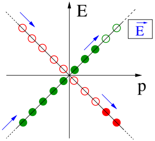


2. Add electric field to a single right/left-mover:



Nothing happens to single states, except energy change.

3. However, for *infinite mode sums* the level shifts induce a surplace of right-movers:



$\dot{Q}_V = \dot{Q}_R + \dot{Q}_L = 0$

(vector charge still conserved)

$\dot{Q}_A = \dot{Q}_R - \dot{Q}_L = \frac{L}{2\pi} 2eE \neq 0$

(axial charge not conserved: anomaly)

$\hookrightarrow$ local & Lorentz inv.: $\partial_\mu j^\mu_A = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$

Level shift method

H.B. Nielsen & M. Ninomiya, PLB 130 (1983)

Dispersion of massless fermions coupled to an uniform electric field in

$A_0 = 0$ gauge : $\quad \partial_0 A_1 = E \quad$ and $\quad \mathrm{i}\partial_0 \psi_{R/L}(x) = \pm(-\mathrm{i}\partial_1 + eA_1)\psi_{R/L}(x)$

- Right/left-handed fermions: $\omega_{R/L} = \pm(p + eA_1)$ and $\dot{\omega}_{R/L} = \pm eE$
- Quantization in a box of length $L \rightsquigarrow$ density of states $\frac{\Delta N}{\Delta p} = \frac{L}{2\pi}$

$$\begin{aligned} \dot{N}_{R/L} &= \frac{\Delta N_{R/L}}{\Delta t} = \frac{\Delta N_{R/L}}{\Delta p_{R/L}} \frac{\Delta p_{R/L}}{\Delta t} \\ &= \frac{L}{2\pi}\,\dot{\omega}^{\mathrm{Fermi}}_{R/L} = \frac{L}{2\pi}(\pm eE) = \dot{Q}_{R/L} \end{aligned}$$

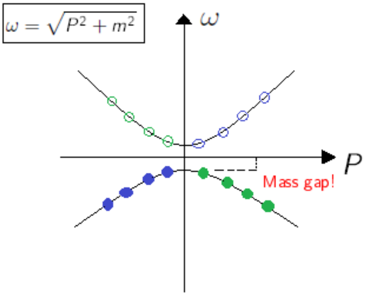
Vector charge

$\dot{Q}_V = \dot{N}_R + \dot{N}_L = 0 \leftarrow$ gauge invariance

Axial charge

$\dot{Q}_A = \dot{N}_R - \dot{N}_L = \frac{Le}{\pi}E \leftarrow$ anomaly

Explicit symmetry breaking: modification due to a mass term

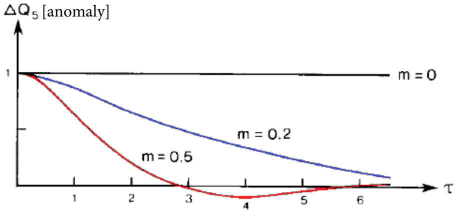


Vector potential

$$A_1(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ Et & \text{"} \quad 0 \leq t \leq \tau \\ E\tau & \text{"} \quad t \geq \tau \end{cases}$$

If E is turned on, two limiting cases:

1. Diabatic approximation: $\tau^{-1} \gg m$
   E suddenly turned on $\rightsquigarrow \Delta Q_A \neq 0$
2. Adiabatic approximation:
   $\tau^{-1} \ll m$
   E slowly turned on $\rightsquigarrow \Delta Q_A = 0$



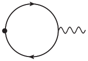
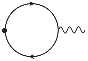
J. Ambjørn et al., NP B221 (1983) 381 (note: m in units of $[t^{-1}]$)

Mass modification (II) – Pauli-Villars regularization

W. Pauli & F. Villars, Rev. Mod. Phys. 21 (1949):

Ghost-fermions with Bose statistics and mass $M \neq 0$ added **gauge invariantly** to standard Lagrangian; limit $M \to \infty$ assumed in the end:

$$iq_\mu J_5^\mu(q) = iq_\mu\, \bar\psi(q)\gamma^\mu\gamma^5\psi(q) = 2im\,\bar\psi(q)\gamma^5\psi(q)$$

$\longrightarrow iq_\mu$  $-\ iq_\mu$ 

m $\qquad$ M

$$= (-)ie\int\frac{d^2k_\ell}{(2\pi)^2}\,\mathrm{Tr}\left[2im\gamma^5\frac{1}{\not k_\ell - m}\gamma^\nu\frac{1}{\not k_\ell + \not q - m}\right]$$

$$+\lim_{M\to\infty} ie\int\frac{d^2k_\ell}{(2\pi)^2}\,\mathrm{Tr}\left[2iM\gamma^5\frac{1}{\not k_\ell - M}\gamma^\nu\frac{1}{\not k_\ell + \not q - M}\right]$$

$$\xrightarrow[M\to\infty]{m\to 0}\ 0 - \lim_{M\to\infty} 2Me\int\frac{d^2k_\ell}{(2\pi)^2}\frac{\mathrm{Tr}\left[\gamma^5(\not k_\ell + M)\gamma^\nu(\not k_\ell + \underline{\not q})\right]}{(k_\ell^2 - M^2)^2} = +ie\,\frac{\epsilon^{\mu\nu}}{\pi}q_\mu$$

Note: if also $m \to \infty$, then $\partial_\mu J_5^\mu = 0$ as expected:

$\hookrightarrow$ anomaly contribution completely canceled by the mass term.

Level shift method in 3+1 dimensions

Landau levels

H.B. Nielsen & M. Ninomiya, PLB 130 (1983) 389

1. Weyl fermions in uniform magnetic field H along 3rd axis:

$$A^\mu(\vec{x},t) = \delta^{\mu\,2}\, H\, x^1 \quad \text{in temporal gauge } A_0 \equiv 0$$

2. In the end, a parallel electric field E is switched on as well:

$$\delta A^\mu(\vec{x},t) = \delta^{\mu\,3}\,(-E)\,t \qquad \text{for} \quad 0 < t < \tau$$

- The e.o.m. $\left[i\partial_t \mp (-i\vec{\nabla} - e\vec{A})\cdot\vec{\sigma}\right]\psi_{R/L}(x) = 0$
- **Ansatz:** $\psi_{R/L} = \left[i\partial_t \pm (-i\vec{\nabla} - e\vec{A})\cdot\sigma\right]\phi_{R/L}(x)$ (auxiliary field)
- Satisfies differential eq. of harmonic oscillator type in 2 D plus free motion along the 3rd axis:

$$\omega(n,\sigma_3,p_3) = \pm\sqrt{2eH\left(n+\tfrac{1}{2}\right) + eH\sigma_3 + p_3^2} \quad \text{with the EVs } \sigma_3 = \pm 1$$

$\hookrightarrow$ **Landau levels**

Landau levels and chiral anomaly in 3+1 D

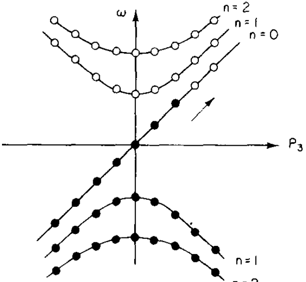
$$\omega(n,\sigma_3,p_3) = \pm\sqrt{2eH\left(n+\tfrac{1}{2}\right) + eH\sigma_3 + p_3^2} \quad \text{with the EVs} \;\; \sigma_3 = \pm 1$$

1. **Double-degenerate hyperbolic** dispersions if $n > 0$ or $\sigma_3 = +1$:

$$\omega(n,\sigma_3\!=\!+1,p_3) = \omega(n\!+\!1,\sigma_3\!=\!-1,p_3) \quad \rightsquigarrow \quad \pm\sqrt{(2n\!+\!1)eH + p_3^2}$$

2. but **non-degenerate linear** dispersions for $n = 0$ and $\sigma_3 = -1$:

$$\omega_{R/L}(n\!=\!0,\sigma_3 = -1,p_3) = \pm p_3$$ $\rightsquigarrow$ right-moving ($\omega_R = +p_3$) & left-moving ($\omega_L = -p_3$)



Nielsen & Ninomiya (1983)

$$\dot N_R = \underbrace{\overbrace{\frac{eH\,L_1 L_2}{2\pi}}^{\text{2D Landau}} \times \overbrace{\frac{L_3}{2\pi}}^{\text{1D}}}_{\text{d.o.s}} \times \overbrace{\dot\omega_R^{\text{Fermi}}(n\!=\!0,\sigma_3\!=\!-1,p_3)}^{+eE \text{ as in 1+1 D}}$$

$$= -\dot N_L \quad \rightsquigarrow \quad \dot Q_V = \dot N_R + \dot N_L = 0$$

$$\& \quad \dot Q_A = \dot N_R - \dot N_L = \frac{L_1 L_2 L_3}{2\pi^2} e^2 EH$$

$$\xrightarrow{\text{Lorentz cov.}} \quad \partial_\mu J_A^\mu = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

Adler-Bell-Jackiw anomaly in 3+1 D

also called abelian chiral anomaly

S. Adler, Phys Rev. 177 (1969) 2426; J.S. Bell & R. Jackiw, Il Nuovo Cim. A60 (1969) 47

In integral form:

$$\dot{Q}_A = \frac{e^2}{8\pi^2}\int d^3x\, 4\,\vec{E}\cdot\vec{B} = -\frac{e^2}{8\pi^2}\int d^3x\, F_{\mu\nu}\tilde{F}^{\mu\nu} \quad \text{with} \quad \tilde{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,.$$

In local operator form (plus expl. breaking due to fermion mass term):

$$\partial_\mu J_A^\mu(\vec{x},t) = -\frac{e^2}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + 2mi\bar{\psi}\gamma_5\psi\,.$$

- **QED case:** valid for dynamical as well as background $\vec{E}, \vec{B}$ fields.
- also valid for non-abelian gauge fields, *e.g.* **QCD case**:

  replace $e^2\, F\tilde{F}$ by $g_s^2 \mathrm{Tr}_{\mathrm{flavor}}(\mathbb{1})\mathrm{Tr}_{\mathrm{color}}\left(t^a t^b\right) G^a\tilde{G}^b = g_s^2 N_F \frac{1}{2} G^a \tilde{G}^a$.
- Should be discriminated from the *non-abelian* chiral anomaly

$$\mathcal{D}_\mu J_A^{a\,\mu} = -\frac{1}{4\pi}\mathrm{Tr}\left[t^a g_s^2\left(\tfrac{1}{4}F_V\tilde{F}_V + \tfrac{1}{12}F_A\tilde{F}_A + \tfrac{2}{3}ig_s\left(AA\tilde{F}_V + A\tilde{F}_V A + \tilde{F}_V AA\right) - \tfrac{8}{3}g_s^2 AA\widetilde{AA}\right) + \ldots\right]$$

Overview about chiral anomaly vs. regularization

Regularization methods have to be invariant under gauge transf's:

- Point-splitting and Wilson line (because of gauge invariance):

$$J_A^\mu \to \lim_{\epsilon\to 0_+} \bar{q}(x+\epsilon/2)\gamma^\mu\gamma^5 e^{-ie\int_{x-\epsilon/2}^{x+\epsilon/2} A^\nu(x')dx'_\nu} q(x-\epsilon/2)$$

- Standard cutoff-regularization: loop integral linearly divergent:
  shift of the integrand $\rightsquigarrow$ extra surface term $\xrightarrow{\text{gauge inv.}}$ anomaly

- Dimensional regularization (which is gauge invariant):
  $$\not{q}\,\gamma^5 = (\not{q} + \not{k}_{\rm loop} - \not{k}_{\rm loop})\gamma^5 = (\not{k}_{\rm loop} + \not{q})\gamma^5 + \gamma^5\,\not{k}_{\rm loop}\ \underbrace{-\,\{\not{k}_{\rm loop},\gamma^5\}}_{\hookrightarrow\ \text{anomaly}}$$

- Pauli-Villars method: anomaly from the presence of the non-zero regulator mass M of the ghost fields (by preserving gauge invariance).

- (Eucl.) Path-integrals $\int[D\psi D\bar{\psi}]\exp(-\int d^4x\,L)$ with $\psi \to \psi' = U\psi$:
  action still invariant, $S[\psi] \longrightarrow S[\psi'] = S[\psi]$, but measure i.g. not:
  $$[D\psi D\bar{\psi}] \xrightarrow{\psi\to\psi'} [D\psi D\bar{\psi}] \times (\text{Jacobian})^{-2} = [D\psi D\bar{\psi}]e^{-i\alpha\int \text{anomaly}_E}$$

The decay of the neutral pion: π^0 (135 MeV, $J^{PC} = 0^{-+}$)

- $\Gamma_{\text{exp.}}(\pi^0 \to \gamma\gamma) = 7.63\,\text{eV}$ and $\mathcal{BR}_{\text{exp.}}(\pi^0 \to \gamma\gamma) = 98.92\%$, such that $\Gamma_{\text{total}}(\pi^\pm)$ ca. $10^{-9}\times$ smaller because $\pi^+ \to \bar{\mu}\nu_\mu$ is a weak decay.
- Apply **Partial Conservation of the Axial Current** (PCAC):

$$\langle 0|J_A^{\mu a}(x)|\pi^b(q)\rangle = iF_\pi q^\mu\,\delta^{ab} e^{-iqx}\,,$$

  and

$$\partial_\mu\langle 0|J_A^{\mu a}(x)|\pi^b(q)\rangle = F_\pi q^2 \delta^{ab} e^{-iqx} = F_\pi M_\pi^2\,\delta^{ab} e^{-iqx}\,.$$

  ($F_\pi = 92.2\,\text{MeV}$ is the pion decay constant and M_π the pion mass)

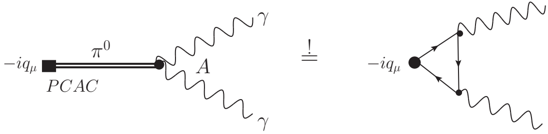
- **In the chiral limit** $M_\pi^2 \to 0$, the axial current seems to be 'conserved'; however, there is still the **anomaly** for the $a = 3$ (π^0) case:

$$\partial_\mu J_A^{\mu 3} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \times \underbrace{\text{Tr}_{\text{color}}(\mathbb{1}) \times \text{Tr}_{\text{flavor}}\left(\frac{1}{2}\tau^3\,\mathbf{Q}^2\right)}_{N_c\times\frac{1}{2}(Q_u^2-Q_d^2) = 3\times\frac{1}{2}\left(\frac{4}{9}-\frac{1}{9}\right) = \frac{1}{2}}$$

π^0 decay via 't Hooft anomaly matching and PCAC

$$\partial_\mu J_A^{\mu 3} = -\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \quad \text{(assuming } N_c = 3\text{)}$$

$$\text{and} \quad \mathcal{M}(\pi^0 \to \gamma\gamma) \equiv A\epsilon^{\rho\sigma\alpha\beta}\epsilon^\star_{1\rho}\epsilon^\star_{2\sigma}k_{1\alpha}k_{2\beta}$$



$$\partial_\mu \langle 0|J_A^{\mu 3}|\pi^3\rangle \cdot iD_{\pi^0}(q^2)\cdot i\mathcal{M}(\pi^0\to\gamma\gamma) \stackrel{!}{=} \text{anomaly}$$

$$\Rightarrow (-iq_\mu)(iF_\pi q^\mu)\cdot\frac{i}{q^2 - M_\pi^2{\downarrow}0}\cdot iA = -F_\pi A \stackrel{!}{=} -\frac{e^2}{4\pi^2}$$

$$\Rightarrow A = \frac{e^2}{4\pi^2 F_\pi}, \quad \Gamma(\pi^0\to\gamma\gamma) = |A|^2\frac{M_\pi^3}{64\pi} = 7.76\,\text{eV} \quad (N_c = 3)$$

$$\text{vs. } \Gamma_{\text{exp.}} = (7.63 \pm 0.16)\,\text{eV}$$

$\rightsquigarrow$ pions have to be Goldstone bosons in the chiral limit

Caveat in counting the number of colors

Electric quark charges $Q_{u,d,s}$ for arbitrary number of colors N_c

O. Bär & U.-J. Wiese, NP B 609 (2001)

$$\begin{aligned}
Q(\pi^\pm) &= Q(u\bar{d}/d\bar{u}) = \pm(Q_u - Q_d) &&= \pm 1\,, \\
Q(K^\pm) &= Q(u\bar{s}/s\bar{u}) = \pm(Q_u - Q_s) &&= \pm 1\,, \\
Q(p/n) &= \tfrac{N_c+1}{2} Q_{u/d} + \tfrac{N_c-1}{2} Q_{d/u} &&= 1/0\,, \text{etc.}
\end{aligned}$$

$$\rightsquigarrow \quad Q_u = \tfrac{1}{2}\left(1 + \tfrac{1}{N_c}\right) \quad \text{and} \quad Q_{d,s} = -\tfrac{1}{2}\left(1 - \tfrac{1}{N_c}\right).$$

$\Longrightarrow \Gamma(\pi^0 \to \gamma\gamma)$ independent of number of colors:

$$\partial^\mu J^3_{5\mu} = \frac{-e^2}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \times \mathrm{Tr}_{\text{color}}(\mathbb{1}) \times \mathrm{Tr}_{\text{flavor}}\left[t^3\,\mathbf{Q}^2\right] \stackrel{!}{=} \frac{-e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$\hookrightarrow N_c \times \frac{1}{2}\left(\frac{4}{9} - \frac{1}{9}\right) = \frac{N_c}{6} \to N_c \times \frac{1}{2}\left(\frac{1+2N_c^{-1}+N_c^{-2}}{4} - \frac{1-2N_c^{-1}+N_c^{-2}}{4}\right) = N_c \times \frac{1}{2N_c} = \boxed{\frac{1}{2}}\ \forall N_c.$$

Similarly for $\eta^8 \to \gamma\gamma$, but not for $\eta^1 \to \gamma\gamma$ or physical $\eta, \eta' \to \gamma\gamma$.

DIGRESSIONS

Digression: scale anomaly and *true* generation of hadron masses

(in the case of pure Yang-Mills theory for simplicity)

$S_{\rm YM} = \int d^4x \, \frac{-1}{4g_0^2} G^a_{\mu\nu} G^{a\,\mu\nu}$, classical Yang-Mills action (g_0 bare coupling):

Digression: scale anomaly and *true* generation of hadron masses

(in the case of pure Yang-Mills theory for simplicity)

$S_{\rm YM} = \int d^4x\, \frac{-1}{4g_0^2} G^a_{\mu\nu} G^{a\,\mu\nu}$, classical Yang-Mills action (g_0 bare coupling):

invariant under $x \to \lambda^{-1} x$ and $A^a_\mu \to \lambda A^a_\mu$ (scale transformations with $\lambda > 0$)

Digression: scale anomaly and *true* generation of hadron masses

(in the case of pure Yang-Mills theory for simplicity)

$S_{\rm YM} = \int d^4x \, \frac{-1}{4g_0^2} G^a_{\mu\nu} G^{a\,\mu\nu}$, classical Yang-Mills action (g_0 bare coupling):

invariant under $x \to \lambda^{-1} x$ and $A^a_\mu \to \lambda A^a_\mu$ (scale transformations with $\lambda > 0$)
generated by the current $J^D_\nu = x^\mu \theta_{\mu\nu}$ with the symmetric and *conserved* energy-momentum tensor $\theta_{\mu\nu} = -\frac{1}{g^2}\left(G^a_{\mu\rho} G^{a\,\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} G^a_{\alpha\beta} G^{a\,\alpha\beta}\right)$.

$$\hookrightarrow \quad \boxed{\partial^\nu J^D_\nu \;=\; \theta^\mu_\mu \;\stackrel{!}{=}\; 0 \;\; \text{classically!}}$$

Digression: scale anomaly and *true* generation of hadron masses

(in the case of pure Yang-Mills theory for simplicity)

$S_{\rm YM} = \int d^4x \, \frac{-1}{4g_0^2} G^a_{\mu\nu} G^{a\,\mu\nu}$, classical Yang-Mills action (g_0 bare coupling):

invariant under $x \to \lambda^{-1} x$ and $A^a_\mu \to \lambda A^a_\mu$ (scale transformations with $\lambda > 0$)
generated by the current $J^D_\nu = x^\mu \theta_{\mu\nu}$ with the symmetric and *conserved*
energy-momentum tensor $\theta_{\mu\nu} = -\frac{1}{g^2}\left(G^a_{\mu\rho} G^{a\,\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} G^a_{\alpha\beta} G^{a\,\alpha\beta}\right)$.

$$\hookrightarrow \quad \boxed{\partial^\nu J^D_\nu = \theta^\mu_\mu \stackrel{!}{=} 0 \ \text{ classically!}}$$

Now dimensional regularization to one-loop order:

$$\delta S_{\rm YM} = \int d^{4-\epsilon}x \left[-\frac{1}{4}\left(\frac{1}{g_s^2} + \frac{\beta_0}{8\pi^2}\frac{1}{\epsilon}\right)(\lambda^\epsilon - 1)\, G^a_{\mu\nu} G^{a\,\mu\nu}\right] \quad (\text{here } \beta_0 = 11 N_c/3)$$

$$\to \int d^4x \ln\lambda \left(-\frac{\beta_0}{32\pi^2} G^a_{\mu\nu} G^{a\,\mu\nu}\right) \longrightarrow \boxed{\theta^\mu_\mu = -\frac{\beta_0}{32\pi^2} G^a_{\mu\nu} G^{a\,\mu\nu}}$$

JÜLICH FORSCHUNGSZENTRUM

Digression: scale anomaly and *true* generation of hadron masses

(in the case of pure Yang-Mills theory for simplicity)

$S_{\rm YM} = \int d^4x\, \frac{-1}{4g_0^2} G^a_{\mu\nu} G^{a\,\mu\nu}$, classical Yang-Mills action (g_0 bare coupling):

invariant under $x \to \lambda^{-1} x$ and $A^a_\mu \to \lambda\, A^a_\mu$ (scale transformations with $\lambda > 0$) generated by the current $J^D_\nu = x^\mu \theta_{\mu\nu}$ with the symmetric and *conserved* energy-momentum tensor $\theta_{\mu\nu} = -\frac{1}{g^2}\left(G^a_{\mu\rho} G^{a\,\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} G^a_{\alpha\beta} G^{a\,\alpha\beta}\right)$.

$$\hookrightarrow \boxed{\partial^\nu J^D_\nu = \theta^\mu_\mu \overset{!}{=} 0 \ \text{classically!}}$$

Now dimensional regularization to one-loop order:

$$\delta S_{\rm YM} = \int d^{4-\epsilon}x \left[-\frac{1}{4}\left(\frac{1}{g_s^2} + \frac{\beta_0}{8\pi^2}\frac{1}{\epsilon}\right)(\lambda^\epsilon - 1)\, G^a_{\mu\nu} G^{a\,\mu\nu}\right] \quad (\text{here } \beta_0 = 11 N_c/3)$$

$$\to \int d^4x \ln\lambda \left(-\frac{\beta_0}{32\pi^2} G^a_{\mu\nu} G^{a\,\mu\nu}\right) \longrightarrow \boxed{\theta^\mu_\mu = -\frac{\beta_0}{32\pi^2} G^a_{\mu\nu} G^{a\,\mu\nu}}$$

$\hookrightarrow$ dim. transmutation: mass scale at quantum level via *scale anomaly*

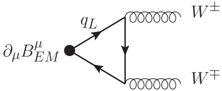
$\hookrightarrow$ in the chiral limit (!) all hadron masses are proportional to this scale

$$0 \neq \langle m_{\rm had}(p) | \theta^\mu_\mu | m_{\rm had}(p') \rangle \overset{p' \to p}{=} g_{\rho\sigma} \left(\frac{p^\rho p^\sigma + p^\sigma p^\rho}{2 m_{\rm had}} \right) = m_{\rm had}(\Lambda_{\rm QCD}) \ (\text{glue ball mass})$$

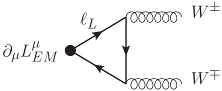
Adler, Collins, Duncan, PRD 15 (1977)

Digression (II): EW Baryogenesis in the Standard Model ?

Conservation of the EM current under weak ($L-R$) interactions:



$+$

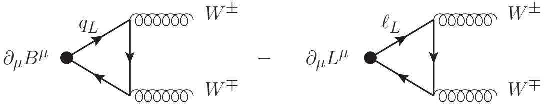


$\propto\; N_c \cdot (Q_u + Q_d) + (0-1) = 1 - 1 = 0$

$\hookrightarrow \Delta(Q_B + Q_L) = 0$ (charge conservation!)

Digression (II): EW Baryogenesis in the Standard Model ?

Conservation of the Baryon−Lepton current under $(L-R)$ interactions:



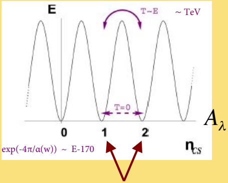
$$\propto N_c \cdot 1/3 - 1 = 1 - 1 = 0$$

$\hookrightarrow \Delta(B-L) = 0$ but $\Delta(B+L) \neq 0$!

Sakharov criteria

1. B violation ✓
   ($\Delta(B+L) \neq 0$ sphaleron transitions)
2. C & CP violation *x*
   (CKM determinant)
3. Nonequilibrium dynamics *x*
   (only fast cross over for $\mu_{chem} = 0$)

Anomalous Processes



Different vacua: $\Delta(B+L) = \Delta N_{CS}$

Sphaleron Transitions

Digression (III): global anomaly and index theorem

In 1+1D:

- For **free fermions** we have $\int d^2x\,\partial_\mu J_A^\mu = \int dt\,\dot{Q}_A = N_R - N_L = 0$
- **Switch on** $F_{01} = E$ such that $\partial_\mu J_A^\mu = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} = \frac{e}{\pi}\partial_\mu(\epsilon^{\mu\nu}A_\nu)$
  - $N_R - N_L$ still vanishes for perturbative gauge transformations respecting periodic boundary conditions at $x = 0, L$.
  - For a constant E, however, we have $N_R - N_L \neq 0$, in fact

$$\int d^2x\,\partial_\mu J_A^\mu = N_R - N_L = \int d^2x\,\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}.$$

    Esp. $N_R - N_L = 2$, *i.e.* the same fermion spectrum as at $t < 0$, but the **right/left-movers are shifted up/down** by one unit, if $A_\nu(x) = \delta_{\nu 1}A_1(t) = (t/\tau)\,2\pi/(eL)$ for $0 < t < \tau$ and zero else.
- This is a 1+1 D version of the Atiyah-Singer index theorem (1963).

Summary of anomaly part of the lecture

- Quantum corrections to a classical Noether current (incuded by a continuous symmetry) might involve **regularizations**
  - which introduce a (regularization) scale
  - and which might not preserve all of the classical symmetries.
- If this is the case, there is a **quantum anomaly:**

  a quantum mechanical obstruction to
  a classical conservation law

- Adler-Bardeen **non-renormalization** theorem: Adler & Bardeen, PR 182 (1969)
  chiral anomaly *is given by lowest-order (1-loop) contribution*
  – proof: either perturbatively or by topological arguments & index theorem
- in 3+1 D: only **triangle anomaly** for abelian gauge fields but triangle, box and pentangle anomalies for non-abelian gauge fields (due to the commutator term in $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig[G_\mu, G_\nu]$)
- all hadron masses in the chiral limit (zero quark masses) result solely from the **scale/trace anomaly** via dimensional transmutation

PART II

$U(1)_A$ problem: why only $N_F^2 - 1$ Pseudo-Goldstone Bosons ?

- GBs arise from **s**pontaneous **s**ymmetry **b**reaking (SSB) with one massless GB per broken symmetry generator (='charge')
- Pseudo-GBs acquire finite mass from small explicit SB
- In the **chiral limit**, the QCD **Lagrangian** is invariant under

$$U(N_F)_L \times U(N_F)_R = \underbrace{SU(N_F)_L \times SU(N_F)_R}_{\text{chiral group}} \times \underbrace{U(1)_V}_{\text{baryon \#}} \times \underbrace{U(1)_A}_{?}$$

$$\xrightarrow{SSB} \{SU(N_F)_L \times SU(N_F)_R / SU(N_F)_V\} \times U(1)_V \times U(1)_A$$

- What about the extra $U(1)_A$ symmetry? Spontaneous SB?
  Is there an extra "(P)GB" in addition to the $N_F^2 - 1$ ones? Not really:

$$\begin{aligned} N_F = 2: &\quad m_{\pi^0} \approx 135\,\text{MeV}\,,\ m_{\pi^\pm} \approx 139\,\text{MeV} \ll m_\eta \approx 548\,\text{MeV} \\ N_F = 3: &\quad m_{\pi^0} \lesssim m_{\pi^\pm} < m_{K^\pm} \lesssim m_{K^0,\bar{K}^0} < m_\eta \ll m_{\eta'} \approx 958\,\text{MeV} \end{aligned}$$

  while for $N_F \geq 2$ there is the naive bound: $m_{"\eta"} \overset{!}{<} \sqrt{3} m_\pi \approx 240\,\text{MeV}$.

  $\hookrightarrow$ This is the $U(1)_A$ problem S. Weinberg, Phys. Rev. D 11 (1975) 3583

- What happens to the extra $U(1)_A$ symmetry at quantum level?

$U(1)_A$ anomaly

- Anomaly of the axial $U(1)_A$ current in **QCD** (in the chiral limit):

$$\partial_\mu J^\mu_A = -\frac{g_s^2 N_F}{16\pi^2} G^a_{\mu\nu}\tilde{G}^{a\,\mu\nu} \qquad \text{(however, } \partial_\mu J^{a\,\mu}_A = 0 \ \forall a \ \text{ since } \mathrm{Tr}_{\text{flavor}}[\tfrac{1}{2}\lambda^a] = 0\text{).}$$

- Under the axial transformation $q_f \to e^{i\alpha\gamma_5/2} q_f$, the chiral anomaly affects the action by (the Jacobian of the path integral measure):

$$-2\ln J = \tfrac{1}{2}\alpha \int d^4x\, \partial_\mu J^\mu_A = -\alpha\,\frac{g_s^2 N_F}{32\pi^2}\int d^4x\, G^a_{\mu\nu}\tilde{G}^{a\,\mu\nu}$$

- Note that $G^a\tilde{G}^a = \partial_\mu K^\mu$ (a total derivative!) with

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(A_\nu G_{\rho\sigma} + i\tfrac{2}{3}A_\nu A_\rho A_\sigma\right) \qquad \text{(Chern-Simons current)}$$

$\hookrightarrow$ the $U(1)_A$ anomaly of QCD is irrelevant in perturbation theory

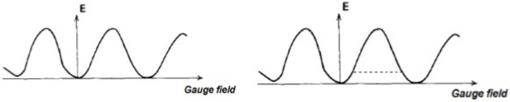
Instantons and non-trivial vacua in QCD

- Non-perturbatively, there exists instantons with

$$\int_{R^4} d^4 x_E \, \frac{1}{32\pi^2} G\tilde{G} = \text{integer} \in \pi_3(G) \quad \text{for } \partial R^4 \equiv S^3 \longrightarrow G \supset SU(2)$$

for a gauge theory with non-Abelian group G with homotopy: $\pi_3(SU(2)) = \mathbb{Z}$

- Thus QCD has a topologically **non-trivial vacuum structure:**



with winding number n

- **instantons** (= *large* gauge transformations) induce $|n\rangle \to |n+1\rangle$ etc.
  $\hookrightarrow$ and solve the $U_A(1)$ problem — 't Hooft, PRL 37 ('76), PRD 14 ('76), 18 ('78)
- However, any **naively chosen vacuum** $|0\rangle_n \equiv |n\rangle$ (n arbitrary, but fixed)
  1. is **unstable** under the one-instanton action, $\Omega_1 : |0\rangle_n \to |0\rangle_{n+1}$,
  2. is **not gauge invariant** under *large* gauge transformations,
  3. ***violates cluster decomposition:*** $\langle O_1 O_2 \rangle \stackrel{!}{=} \langle O_1 \rangle \langle O_2 \rangle$ which can be traced back to causality, unitarity (and locality) of the underlying field theory.

θ vacua in strong interaction physics

Thus true vacuum must be a **superposition of all $|n\rangle$ vacua:**

$$|vac\rangle_\theta \equiv \sum_{n=-\infty}^{+\infty} e^{in\theta}|n\rangle \;\text{ with }\; \Omega_1 : |vac\rangle_\theta \to e^{-i\theta}|vac\rangle_\theta \quad \text{(with a phase shift only)}$$

Note $\;{}_{\theta'}\langle vac|e^{-iHt}|vac\rangle_\theta = \delta_{\theta-\theta'} \times {}_\theta\langle vac|e^{-iHt}|vac\rangle_\theta\;$ such that θ is unique.

$\hookrightarrow$ θ is another parameter of strong interaction physics (as $m_u, m_d, \ldots$):

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^{CP}_{\text{QCD}} + \mathcal{L}^{\cancel{CP}}_{\text{QCD}} = \mathcal{L}^{CP}_{\text{QCD}} + \theta \frac{g_s^2}{32\pi^2} \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} \tilde{G}^a_{\rho\sigma} \,.$$

Under axial rotation of the quark fields $q_f \to e^{i\alpha\gamma_5} q_f \approx (1 + i\alpha\gamma_5) q_f$

$$\mathcal{L}_{\text{QCD}} \to \mathcal{L}^{\text{CP}}_{\text{QCD}} - 2\alpha \textstyle\sum_f m_f \bar{q} i\gamma_5 q + (\theta - 2N_f\alpha) \dfrac{g_s^2}{32\pi^2} \tilde{G}^a_{\mu\nu} G^{a,\mu\nu}$$

$$\hookrightarrow \mathcal{L}^{\text{str}\,\cancel{\text{CP}}}_{\text{SM}} = \mathcal{L}^{\text{CP}}_{\text{SM}} + \bar{\theta} \frac{g_s^2}{32\pi^2} \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma} \quad \text{with } \bar{\theta} = \theta + \arg\det\mathcal{M}$$

Note if any quark mass m_f were zero, then the $\bar{\theta}$ angle could be removed by a suitable axial rotation with $\alpha_f = -\bar{\theta}/2$.

Strong CP problem

The resolution of the $U(1)_A$ problem – via the complicated nature of the QCD vacuum – effectively adds an extra term to the QCD Lagrangian:

$$\mathcal{L}_{\bar\theta} = \bar\theta \frac{g_s^2}{32\pi^2} \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

Strong *CP* problem

The resolution of the $U(1)_A$ problem – via the complicated nature of the QCD vacuum – effectively adds an extra term to the QCD Lagrangian:

$$\mathcal{L}_{\bar\theta} = \bar\theta \frac{g_s^2}{32\pi^2} \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

- This term violates parity P and time-reflection invariance T

  (since only ϵ^{0123} and permutations are non-zero)

  but conserves charge conjugation invariance C $\rightsquigarrow$ $\cancel{CP}$ $\rightsquigarrow$
- it induce an electric dipole moment (EDM) for the neutron:

$$|d_n| \simeq |\bar\theta| \cdot \frac{m_q}{\Lambda_{\rm QCD}} \cdot \frac{e}{2m_n} \sim |\bar\theta| \cdot 10^{-2} \cdot 10^{-14}\, e\,\mathrm{cm} \sim |\bar\theta| \cdot 10^{-16}\, e\,\mathrm{cm}$$

compared with $|d_n^{\rm exp.}| < 2.9 \cdot 10^{-26}\, e\,\mathrm{cm}$ Baker et al., PRL 97 (2006).

$\hookrightarrow$ $|\theta| \lesssim 10^{-10}$, while NDA (naive dim. analysis) predicts $|\bar\theta| \sim \mathcal{O}(1)$.

(Note that the other *CP*-violating phase of the SM, δ_{KM}, is indeed of $\mathcal{O}(1)$).

This mismatch is called **the strong *CP* problem.**

$\mathcal{END\ OF\ THE\ ACTUAL\ PRESENTATION}$

Resolution(s) of the Strong *CP* problem

- **Fine-tuning**
  – motivated by many-worlds scenarios, anthropic principle (?) etc.
- or **spontaneously broken** *CP* such that $\bar{\theta} := 0$ at the Lagrangian level
  – but $\bar{\theta} \neq 0$ reintroduced at the loop level
  – and the CKM mechanism predicts $\not{CP}$ of *explicit* and *not* of SSB nature
- or an **additional chiral symmetry**
  (i) by a **vanishing** (*u*-)quark mass (?)
  – excluded by Lattice QCD: $m_u = 2.7^{+0.7}_{-0.5}$ MeV — Particle Data Group (2014)
  (ii) or by an **additional global** chiral $U_{\mathrm{PQ}}(1)$ symmetry of the SM
  – Peccei-Quinn (PQ) mechanism — Peccei & Quinn, PRL 38 & PRD 16 (1977)
  – including *axions* — Weinberg, PRL 40 (1978), Wilczek PRL (40) (1978)
  ⋮
  (iii) however, the *"Empire strikes back":* fine-tuning *may be back*
  – reintroduced by Planck-scale explicit PQ-symmetry breaking terms.

Kiwoon Choi (Daejeon, Korea), Bethe-Lectures, Bonn, March 2015

Peccei-Quinn symmetry and the axion

Peccei & Quinn (1977): imposed on the SM
a global chiral $U(1)_{\mathrm{PQ}}$ symmetry that is non-linearly realized (*i.e.* SSB)

Peccei & Quinn, PRL 38 & PRD 16 (1977)

Weinberg & Wilczek (1978): introduced the corresponding Nambu–Goldstone boson, the so-called axion Weinberg, PRL 40 (1978), Wilczek PRL (40) (1978)

The static angular parameter $\bar\theta$ (mod 2π) is replaced by a dynamical pseudoscalar field which transforms under PQ as

$$U(1)_{\mathrm{PQ}}: \; a(x) \to a(x) + f_a \alpha_{\mathrm{PQ}}$$

where f_a is the order parameter associated with SB $U(1)_{PQ}$ symmetry.

The SM Lagrangian is augmented by axion interactions

$$\mathcal{L}_{\mathrm{total}} = \mathcal{L}_{\mathrm{SM}} + \bar\theta \frac{g_s^2}{32\pi^2} G^c_{\mu\nu} \tilde G^{c\,\mu\nu} + \frac{1}{2}\partial_\mu a \partial^\mu a + \mathcal{L}_{\mathrm{int}}[\partial^\mu a/f_a, \psi, \bar\psi] + \xi \frac{a}{f_a} \frac{g_s^2}{32\pi^2} G^c_{\mu\nu} \tilde G^{c\,\mu\nu}$$

$\hookrightarrow$ the PQ current is anomalous: $\partial_\mu J^\mu_{PQ} \equiv \partial_\mu \left(f_a \partial^\mu a + \frac{\partial \mathcal{L}_{\mathrm{int}}}{\partial\, \partial_\mu a} \right) = \xi \frac{g_s^2}{32\pi^2} G^c_{\mu\nu} \tilde G^{c\,\mu\nu}$

The effective potential for the axion field

The minimum of this effective potential occurs at $\langle a \rangle = -\bar{\theta} f_a/\xi$:

$$\left\langle \frac{\partial V_{\rm eff}}{\partial a} \right\rangle = -\frac{\xi}{f_a} \left\langle \frac{g_s^2}{32\pi^2} G^c_{\mu\nu} \tilde{G}^{c\,\mu\nu} \right\rangle \Bigg|_{\langle a \rangle = -\bar{\theta} f_a/\xi} = 0$$

such that the $\bar{\theta}$ term is canceled out at this minimum.

Without the QCD anomaly, the $U(1)_{\rm PQ}$ symmetry is compatible with all values

$$0 \le \xi \frac{\langle a \rangle}{f_a} < 2\pi \,.$$

With the QCD anomaly the axion potential has to be periodic in the *effective* vacuum angle $\bar{\theta} + \langle a \rangle \xi / f_a$:

$$V_{\rm eff} \sim -\cos\left(\bar{\theta} + \xi \frac{\langle a \rangle + a_{\rm phys.}}{f_a} \right) \qquad \text{with the minimum at} \quad \langle a \rangle = -\frac{f_a}{\xi} \bar{\theta}$$

$$\text{and} \quad m_a^2 = \left\langle \frac{\partial^2 V_{\rm eff}}{\partial a^2} \right\rangle = -\frac{\xi}{f_a} \frac{\partial}{\partial a} \left\langle \frac{g_s^2}{32\pi^2} G^c_{\mu\nu} \tilde{G}^{c\,\mu\nu} \right\rangle \Bigg|_{\langle a \rangle = -\bar{\theta} f_a/\xi} \quad \text{as axion mass}$$

Low-energy effective field theory

For $p < \Lambda_{QCD}$: low-energy EFT of $\Sigma_{ij} = \left(e^{i\phi}\right)_{ij} \propto \langle \bar{q}_{Rj} q_{Li} \rangle$ with ϕ nonet of pseudoscalar mesons: π, K, η, η'

$U(3)_L \times U(3)_R$ reparameterization :

$$\Sigma \to L\Sigma R^\dagger, \quad M \to LMR^\dagger, \quad \bar{\theta} \to \bar{\theta} - \arg\det(LR^\dagger)$$

$$V(\Sigma) = \underbrace{V_0\left(e^{i\bar{\theta}}\det\Sigma\right)}_{\sim\Lambda^4_{\text{QCD}}} - \underbrace{v^3\,\mathrm{Tr}\left(M\Sigma^\dagger + \text{h.c.}\right)}_{\sim\Lambda^3_{\text{QCD}} M \;\;\&\;\; v_3 := \frac{m_\pi^2 f_\pi^2}{(m_u+m_d)}}$$

$\bar{\theta} = \xi\langle a\rangle/f_a$ for $a \to a + 2\pi f_a$ (axion periodicity) with axion decay constant f_a and $\xi = N_{\text{DW}}$ (multiplicity of axion vacua)

$$V(a) = V_{\bar{\theta}=\xi a/f_a} = -f_\pi^2 m_\pi^2 \sqrt{\frac{m_u^2 + m_d^2 + 2m_u m_d \cos(N_{\text{DW}} a/f_a)}{(m_u+m_d)^2}} \equiv V_{\text{QCD}}(a)$$

Dyn. relaxation: $\bar{\theta} = \xi\frac{\langle a\rangle}{f_a}$, $\quad m_a{}^2 = \frac{\partial^2}{\partial a^2} V(a)\big|_{\langle a\rangle=0} \approx \frac{m_u m_d}{(m_u+m_d)^2}\frac{f_\pi^2 m_\pi^2}{(f_a/\xi)^2}$

The road to the invisible axion models

$$\mathcal{L}_{\text{axion-ints}} = \mathcal{L}_{\text{int}}(\partial^\mu a/f_a, \psi_f) + \xi \frac{a}{f_a} \frac{g_s^2}{32\pi^2} G^c_{\mu\nu} \tilde{G}^{c\,\mu\nu}$$

The $U(1)_{\mathrm{PQ}}$ order parameter f_a:

- f_a associated with the scale of the spont. breaking of the PQ symm.
- original PQ-model (w. two Higgs) had $f_a \sim v_F \equiv \sqrt{v_1^2 + v_2^2} \approx 246\,\mathrm{GeV}$ and predicted $\mathcal{BR}(K^+ \to \pi^+ + a) < 3 \cdot 10^{-5} \cdot (v_2/v_1 + v_1/v_2)$
- however $\mathcal{BR}_{\text{exp}}(K^+ \to \pi^+ \text{nothing}) < 3.8 \cdot 10^{-8}$ Asano et al. (KEK), PLB 107 (1981) such that $f_a \gg v_F$
- Basically two classes of invisible axion models:

  KSVZ model: scalar field σ with $f_a = \langle\sigma\rangle \gg v_F$ and super-heavy quark with PQ charge and $M_Q \sim f_a$ Kim, PRL 43 ('79); Shifman, Vainshtein, Zakharov, NPB 166 ('80)

  DFSZ model: adds to original PQ model a scalar field with PQ charge and $f_a = \langle\phi\rangle \gg v_F$ Dine, Fischler, Srednicki, PLB 104 ('81), Zhitnitsky, Sov.J.NP 31 ('80)

Photon couplings to axions

QCD anomaly induces an anomalous axion-coupling to 2 photons, *e.g.:*

$$\mathcal{L}^{\mathrm{KSVZ}}_{\mathrm{axion}} = \frac{a}{f_a}\left(\xi \frac{g_s^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\,\mu\nu} + 3e_Q^2 \frac{\alpha_{\mathrm{EM}}}{4\pi} F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

The $a\gamma\gamma$ coupling has to be corrected by the axion mixing with the lowest pseudoscalars:

$$3e_Q^2 \to 3e_Q^2 - \frac{4m_d + m_u}{3(m_u + m_d)}$$

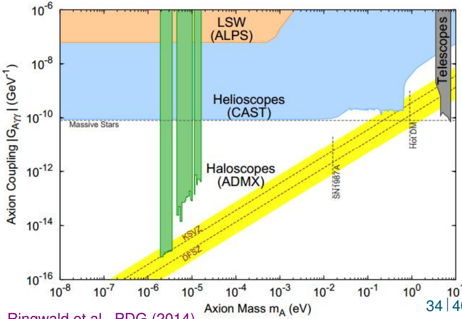
$a\gamma\gamma$ couplings in general:

$$G_{a\gamma\gamma} = \frac{\alpha_{\mathrm{EM}}}{4\pi f_a}\left[\frac{2E}{N} - \frac{4m_d + m_u}{3(m_d + m_u)}\right]$$

E & N strength of em / strong anomaly, respectively:

DSFZ: $E/N = 8/3$,

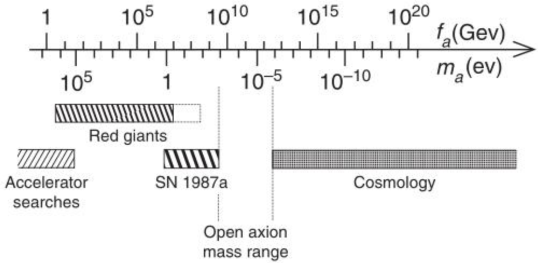
KSVZ: $E/N = 0$ (if $e_Q := 0$)



A. Ringwald et al., PDG (2014)

Window for axion searches

S.Asztalos, ed. G.Bertone, Cambridge Univ. Press ('10)



Axion searches

- in Labs (Colliders, Lasers) – light-shining-through-the-wall (*e.g.* ALPSII)
- for Astro-sources –Helioscopes (*e.g.* CAST, IAXO)
- for galactic axions – Halioscopes/microwave cavities (*e.g.* ADMX)
- indirect constraints: – from Astrophysics (red giants, SN 1987a)
  – and from Cosmology: DM bounds ($\Omega_{CDM} \approx 0.22$) on axion oscillations

$$\rightsquigarrow \quad f_a^{max} \leftrightarrow m_a^{min}$$

Preliminary Summary: Axions

- predicted as a / the resolution of the *Strong CP problem*:
  to escape the fine-tuning problem $|\bar{\theta}| < 10^{-10}$ while $\delta_{\rm KM} \sim \mathcal{O}(1)$
- extendible to ALPS: axion-like particles with f_a and $g_{\rm alps\gamma\gamma}$ decoupled
- couple feebly ($\sim 1/f_a$) and gravitationally to matter and radiation
- can be candidates for Cold Dark Matter
  i.e. with a well-determined and narrow window for searches:

however fine-tuning may back ...

THE EMPIRE STRIKES BACK

Axions and EDMs: generic effective Lagrangian of the axion

Kiwoon Choi (Daejeon, Korea), Bethe-Lectures, Bonn, March 2015

$$\mathcal{L}_{\rm eff}(a) = \underbrace{\mathcal{L}_0}_{\text{indep. of } a} + \underbrace{\tfrac{1}{2}(\partial_\mu a)^2 + \frac{\partial_\mu a}{f_a}\tilde{J}^\mu(\bar\psi...\psi,\phi)}_{\text{PQ-invariant}} + \underbrace{\frac{a}{f_a}\frac{N}{32\pi^2}G\tilde{G}}_{\text{expl. PQ-breaking by QCD anomaly}}$$

$$+\underbrace{\Delta\mathcal{L}_{UV}\left(= -\epsilon m_{\rm UV}^4\cos(a/f_a+\delta_{\rm UV})\right)}_{a \text{ coupling from expl. PQ breaking at UV scale}}$$

$\bar\theta = \langle a\rangle/f_a$ is calculable in terms of the $\cancel{CP}$ angles (in presence of a!):

- $\delta_{\rm KM}$ = Kobayashi-Maskawa phase in the PQ-invariant SM
- $\delta_{\rm BSM}$ = $\cancel{CP}$ phase in PQ-invariant BSM at the scale $m_{\rm BSM}$
- $\delta_{\rm UV}$ = $\cancel{CP}$ phase in explicit PQ-breaking sector at $m_{\rm UV} \sim M_{\rm Planck}$

$$V_{\rm QCD} \sim f_\pi^2 m_\pi^2\cos(a/f_a) \quad \text{(expl. PQ-breaking by low-energy QCD)}$$

$$V_{\rm KM} \sim f_\pi^2 m_\pi^2 \times \overbrace{G_F^2 f_\pi^4}^{10^{-14}} \times \overbrace{10^{-5}\sin\delta_{\rm KM}}^{\text{Jarlskog inv.}} \times \sin(a/f_a)$$

$$V_{\rm BSM} \sim f_\pi^2 m_\pi^2 \times \underbrace{\left(10^{-2}-10^{-3}\right)}_{\text{loop suppression}} \times \frac{f_\pi^2}{m_{\rm BSM}^2}\sin\delta_{\rm BSM} \times \sin(a/f_a)$$

$$V_{\rm UV} \sim \epsilon m_{\rm UV}^4 \sin\delta_{\rm UV}\sin(a/f_a)$$

$\bar\theta = \langle a\rangle / f_a$ and contributions to the nucleon EDM

$$\bar\theta \;\sim\; 10^{-19}\sin\delta_{\rm KM} + \overbrace{\left(10^{-10}-10^{-11}\right)}^{(10^{-2}\text{–}10^{-3})\times f_\pi^2/\text{TeV}^2} \times \left(\frac{\text{TeV}}{m_{\rm BSM}}\right)^2 \sin\delta_{\rm BSM}$$

$$+\;\epsilon\,\frac{m_{\rm UV}^4}{f_\pi^2 m_\pi^2}\,\sin\delta_{\rm UV} \quad \left(\text{with } \epsilon < 10^{-10}\, f_\pi^2 m_\pi^2/m_{\rm UV}^4 \sim 10^{-88} \text{ for } m_{\rm UV}\sim M_{\rm Pl}\right)$$

$\hookrightarrow$ Regardless of the existence of BSM physics near the TeV scale, $\bar\theta = \langle a\rangle/f_a$ can have *any value* below the present bound $\sim 10^{-10}$.

$$d_N \;\sim\; \frac{e}{m_N}\left[\frac{m^*}{m_N}\bar\theta + G_F^2 f_\pi^4 \times 10^{-5}\sin\delta_{\rm KM} + \left(10^{-2}-10^{-3}\right)\times\frac{f_\pi^2}{m_{\rm BSM}^2}\sin\delta_{\rm BSM} + \left(10^{-2}-10^{-3}\right)\times\frac{f_\pi^2}{m_{\rm UV}^2}\sin\delta_{\rm UV}\right]$$

$$\sim\; \frac{e}{m_N}\left[\overbrace{\frac{m^*}{m_N}}^{\sim 10^{-2}} \times \overbrace{\bar\theta_{\rm UV}}^{\frac{\epsilon m_{\rm UV}^4 \sin\delta_{\rm UV}}{f_\pi^2 m_\pi^2}} + \left(10^{-2}-10^{-3}\right)\times\frac{f_\pi^2}{m_{\rm BSM}^2}\sin\delta_{\rm BSM}\right]$$

likely dominated by $\bar\theta_{\rm UV}$ induced by $\not{CP}$ in the $\not{PQ}$ sector @ $m_{\rm UV}(\sim M_{\rm Pl})$, and/or by the BSM contribution near the TeV scale.

Summary of Part II: from the $U(1)_A$ problem to axions

- $U(1)_A$ problem: $m_{\eta,\eta'} > \sqrt{3}m_\pi \approx 240\,\text{MeV}$
  – solution: complicated QCD vacuum including instantons
- Problem: $|n\rangle$ not unique, not gauge inv., cluster decomposition viol.
  – solution: θ vacuum (superposition of all $|n\rangle$ vacua $\times e^{i\theta n}$)
- Problem: neutron EDM bound $\rightsquigarrow$ strong CP problem
  – solution: Peccei-Quinn mechanism and axions
- Problem: original Peccei-Quinn model w. $f_a = v_F$ excluded by exp.
  – solution: invisible axions with $f_a \gg v_F$
- Problem: how to detect an (invisible) axion
  – possible solution: direct/indirect searches in rather narrow window
- Problem: fine-tuning back from explicit PQ-breaking at the UV scale
  – possible solution: check several EDMs (*e.g.* $d_n, d_p, d_D, d_{^3\text{He}}, \ldots$)

Jump slides

The symmetries of QCD

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{2}\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) + \sum_f \bar{q}_f(\mathrm{i}\not{D} - m_f)q_f + \ldots$$

$$D_\mu = \partial_\mu - igA_\mu \equiv \partial_\mu - igA_\mu^a\frac{\lambda^a}{2}\,, \qquad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

- Lorentz-invariance, P, C, T invariance, $\mathrm{SU}(3)_c$ gauge invariance
- The masses of the u, d, s quarks are small: $m_{u,d,s} \ll 1\,\mathrm{GeV} \approx \Lambda_{\rm hadron}$.
- Chiral decomposition of quark fields:

$$q = \tfrac{1}{2}(1-\gamma_5)q + \tfrac{1}{2}(1+\gamma_5)q = q_L + q_R\,.$$

- For massless fermions: left-/right-handed fields do not interact

$$\mathcal{L}[q_L, q_R] = i\bar{q}_L\not{D}q_L + i\bar{q}_R\not{D}q_R - m\,(\bar{q}_Lq_R + \bar{q}_Rq_L)$$

and $\mathcal{L}^0_{QCD}$ invariant under (global) chiral $\mathrm{U}(3)_L\times\mathrm{U}(3)_R$ transformations:

$\hookrightarrow$ rewrite $\mathrm{U}(3)_L\times\mathrm{U}(3)_R = \mathrm{SU}(3)_V\times\mathrm{SU}(3)_A\times\mathrm{U}(1)_V\times\mathrm{U}(1)_A$.

- $\mathrm{SU}(3)_V = \mathrm{SU}(3)_{R+L}$: still conserved for $m_u = m_d = m_s > 0$
- $\mathrm{U}(1)_V = \mathrm{U}(1)_{R+L}$: quark or baryon number is conserved
- $\mathrm{U}(1)_A = \mathrm{U}(1)_{R-L}$: broken by quantum effects ($\mathrm{U}(1)_A$ anomaly + instantons)

JÜLICH FORSCHUNGSZENTRUM

Hidden Symmetry and Goldstone Bosons

$$[Q_V^a, H] = 0, \quad \text{and} \quad e^{-iQ_V^a}|0\rangle = |0\rangle \Leftrightarrow Q_V^a|0\rangle = 0 \quad \text{(Wigner-Weyl realization)}$$

$$[Q_A^a, H] = 0, \quad \text{but} \quad e^{-iQ_A^a}|0\rangle \neq |0\rangle \Leftrightarrow Q_A^a|0\rangle \neq 0 \quad \text{(Nambu-Goldstone realiz.)}$$

- Consequence: $e^{-iQ_A^a}|0\rangle \neq |0\rangle$ is not the vacuum, but

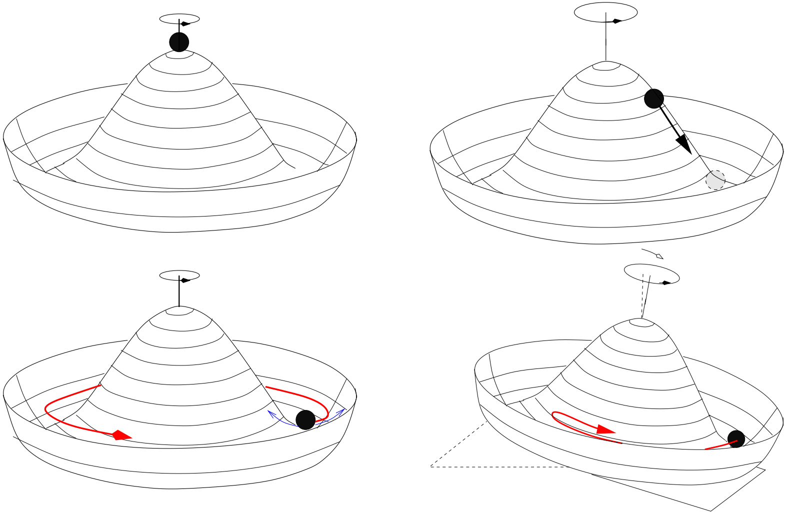
$$H\, e^{-iQ_A^a}|0\rangle = e^{-iQ_A^a} H|0\rangle = 0 \quad \text{is a massless state!}$$

Goldstone theorem: *continuous* global symmetry that does *not* leave the ground state invariant ('hidden' or 'spontaneously broken' symm.)

- mass- and spinless particles, "Goldstone bosons" (GBs)
- number of GBs = number of broken symmetry generators
- axial generators broken $\Rightarrow$ GBs should be pseudoscalars
- finite masses via (small) quark masses
  $\hookrightarrow$ 8 lightest hadrons: $\pi^\pm$, π^0, $K^\pm$, K^0, $\bar{K}^0$, η (not η')
- Goldstone bosons decouple (non-interacting) at vanishing energy

back1 back2

Illustration: spontaneous symmetry breaking (SSB)



back1 back2

Decoupling theorem of Goldstone bosons

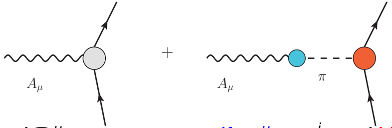
Goldstone bosons do not interact at zero energy/momentum

1. $Q^a_A|0\rangle \neq 0 \quad \Rightarrow \quad Q^a_A$ creates GB $\quad \Rightarrow \quad \langle \pi^a | Q^a_A | 0\rangle \neq 0.$
2. Lorentz invariance $\rightsquigarrow \langle \pi^a(q)|A^\mu_b(x)|0\rangle = -if_\pi\, q^\mu\, \delta^a_b\, e^{iq\cdot x} \neq 0\,!$

   A^μ_b axial current

   $\hookrightarrow f_\pi \neq 0$ necessary for SSB (order parameter)

   (pion decay constant $f_\pi = 92\,\text{MeV}$ from weak decay $\pi^+ \to \mu^+ \nu_\mu$)
3. Coupling of axial current A_μ to matter fields (and/or pions)

$i\mathcal{A}^\mu =$ 

$$= \quad i\,\mathcal{R}^\mu \text{ (non-sing.)} \quad + \quad -if_\pi\, q^\mu \frac{i}{q^2 - m_\pi^2 + i\epsilon}\, i\,V \quad (V\text{: coupling of GB to matter fields})$$

4. Conservation of axial current $\partial_\mu A^\mu_b(x) = 0$: $\Rightarrow m_\pi^2 = 0$ and $q_\mu \mathcal{A}^\mu = 0$:

$$0 = q_\mu \mathcal{R}^\mu - f_\pi \frac{q^2}{q^2} V \xrightarrow{q\to 0} 0 = -f_\pi \lim_{q\to 0} V \xrightarrow{f_\pi \neq 0} \lim_{q\to 0} V = 0 \quad \Rightarrow \quad \text{decoupling!}$$

Degeneracy of Landau levels

The density of states for Landau levels reads:

$$\Delta n = \underbrace{\left[L_2 \int_0^{k_2^{\max}=L_1 eH} \frac{dk_2}{2\pi} \right]}_{\substack{\text{\# of states in square } L_1 \cdot L_2 \perp \vec{B}, \\ 0 \le x_1 + \frac{k_2}{eH} \le L_1 \;\Rightarrow\; k_2^{\max} = L_1 eH}} \cdot \underbrace{L_3 \frac{\Delta k_3}{2\pi}}_{\frac{1}{h}\int dx\, dp}$$

$$= L_2 \int_0^{L_1 eH} \frac{dk_2}{2\pi} L_3 \frac{\Delta k_3}{2\pi}$$

$$= \frac{L_1 L_2 eH}{2\pi} \frac{L_3 \Delta k_3}{2\pi}$$

back

CP violation and the Electric Dipole Moment (EDM)

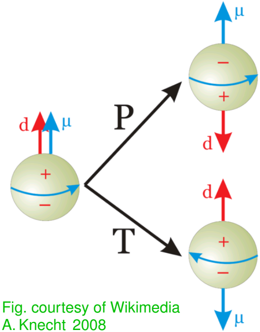


Fig. courtesy of Wikimedia
A. Knecht 2008

EDM: $\vec{d} = \sum_i \vec{r}_i\, e_i \xrightarrow[\text{particles}]{\text{subatomic}} d \cdot \vec{S}/|\vec{S}|$

(polar) (axial)

$$\mathcal{H} = -\mu \frac{\vec{S}}{S} \cdot \vec{B} - d \frac{\vec{S}}{S} \cdot \vec{E}$$

P: $$\mathcal{H} = -\mu \frac{\vec{S}}{S} \cdot \vec{B} + d \frac{\vec{S}}{S} \cdot \vec{E}$$

T: $$\mathcal{H} = -\mu \frac{\vec{S}}{S} \cdot \vec{B} + d \frac{\vec{S}}{S} \cdot \vec{E}$$

Any *non-vanishing EDM* of a non-deg. (subatomic) particle violates **P** & **T**

- Assuming CPT to hold, CP is violated as well
  ↪ subatomic EDMs: “rear window” to CP violation in early universe
- Strongly suppressed in SM (CKM-matrix): $|d_n| \sim 10^{-31} e\,\mathrm{cm}$, $|d_e| \sim 10^{-38} e\,\mathrm{cm}$
- Current bounds: $|d_n| < 3 \cdot 10^{-26} e\,\mathrm{cm}$, $|d_p| < 8 \cdot 10^{-25} e\,\mathrm{cm}$, $|d_e| < 1 \cdot 10^{-28} e\,\mathrm{cm}$

  n: Baker et al. (2006), *p* prediction: Dimitriev & Sen’kov (2003)*, *e*: Baron et al.(2013)†

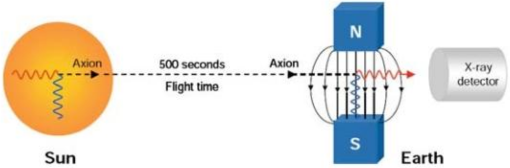
  * from $|d_{^{199}\mathrm{Hg}}| < 3.1 \cdot 10^{-29} e\,\mathrm{cm}$ bound of Griffith et al. (2009) † from polar ThO: $|d_{ThO}| \lesssim 10^{-21} e\,\mathrm{cm}$

Instanton amplitudes

- Since $G^a_{\mu\nu}\tilde{G}^{a\,\mu\nu} = \partial_\mu K^\mu$ is a total derivative,
$$\mathcal{L}_{\rm QCD} = \bar\theta \frac{g_s^2}{32\pi^2}\,\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$
is irrelevant in perturbation theory.
- Non-perturbatively, large gauge transformations (instantons) exist:
$$\int_{R^4} d^4x_E \,\frac{1}{32\pi^2} G\tilde{G} \;=\; \text{integer}$$
- Some amplitudes depend on the periodic angle parameter
$$\bar\theta = \bar\theta + 2\pi \,:$$
$$\mathcal{A}_\theta \propto e^{-S_E} \propto e^{-\int d^4x_E\left(\frac{1}{8g_s^2}(G\pm\tilde{G})^2 \mp \left(\frac{8\pi^2}{g_s^2}\mp i\bar\theta\right)\frac{1}{32\pi^2}G\tilde{G}\right)} \propto e^{-\frac{8\pi^2}{g_s^2(\mu)}\pm i\bar\theta}$$
- Weak coupling $g_s^2(\mu) \ll 1$: instanton amplitudes exponentially small.
- For strong coupling $g_s^2(\mu) \sim 8\pi^2$, no suppression

Helioscopy

R. Battesti et al., Springer Lect. Notes Phys. 741 (2008)



- Time-reversed *Primakoff effect*: $a + \gamma_{\text{virtual}} \to \gamma$
- most sensitive for $10^{-5}\,\text{eV} \le m_a \le 1\,\text{eV}$
- depends on field B, length L, transferred momentum $q = m_a/2E$
- and solar models

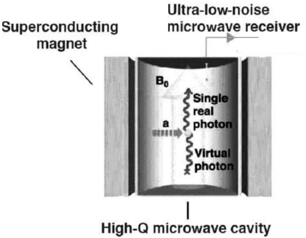
CAST experiment (CERN Axion Solar Telescope)

- $m_a < 1.17\,\text{eV}$ (intersecting the KSVZ band)
- Next generation: IAXO (International Axion Oberservatory)@CERN

back

Halioscopy

R. Battesti et al., Springer Lect. Notes Phys. 741 (2008)



- Search for galactic axions via Primakoff effect: $a + \gamma_{\text{virtual}} \to \gamma$
- Tunable cavity search for microwave resonances
- Most sensitive detectors for CDM axions (μeV $\lesssim m_a \lesssim$ meV)

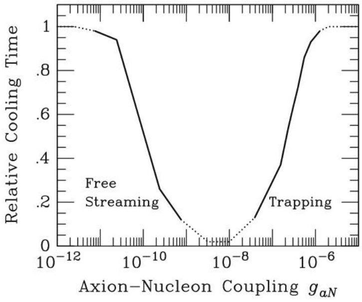
ADMX (Axion Dark Matter eXperiment) @University of Washington

- sensitivity to KVSZ axions between 1.9 μeV $\lesssim m_a \lesssim$ 3.3, μeV
- still on-going (ADMX II)

back

Supernovae (SN1987a)

G.G. Raffelt, Springer Lect. Notes Phys. 741 (2008)



- Axions emitted by nucleon Bremsstrahlung $NN \to NNa$

  – depends therefore on g_{aNN}

- Constraints:
  - energy loss rate $\epsilon_{\text{axion}} \lesssim 10^{19} \text{erg}\, \text{g}^{-1} \text{s}^{-1}$
  - Neutrino burst duration

back