Spontaneous Symmetry Breaking in QCD and Chiral Dynamics
Exercises

1. Consider the QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{2}\mathrm{Tr}_c(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}),$$

   where

$$\begin{aligned} D_\mu q_f &\equiv (\partial_\mu + ig_3\mathcal{A}_\mu) q_f, \\ \mathcal{G}_{\mu\nu} &\equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + ig_3[\mathcal{A}_\mu, \mathcal{A}_\nu], \end{aligned}$$

   and

$$\mathcal{A}_\mu \equiv \sum_{a=1}^{8} \mathcal{A}_{a\mu}\frac{\lambda_a^c}{2}.$$

   The SU(3) gauge transformations of the second kind are defined as

$$\begin{aligned} q_f &\mapsto q_f' = \exp\left[-i\sum_{a=1}^{8}\Theta_a(x)\frac{\lambda_a^c}{2}\right] q_f = U[\Theta(x)] q_f, \\ \mathcal{A}_\mu &\mapsto U\mathcal{A}_\mu U^\dagger + \frac{i}{g_3}\partial_\mu U U^\dagger. \end{aligned}$$

   (a) Show that $D_\mu q_f$ transforms as q_f.

   (b) Verify $\mathcal{G}_{\mu\nu} \mapsto U\mathcal{G}_{\mu\nu}U^\dagger$ Hint: $U^\dagger U = 1$.

   (c) Verify that $\mathcal{L}_{\rm QCD}$ is invariant under the gauge transformations of the second kind.

2. Consider the projection operators

$$P_R = \frac{1}{2}(\mathbb{1} + \gamma_5), \quad P_L = \frac{1}{2}(\mathbb{1} - \gamma_5),$$

   where the indices R and L refer to right-handed and left-handed, respectively. Using the properties

$$\gamma_5^\dagger = \gamma_5, \quad \gamma_5^2 = \mathbb{1},$$

   verify the following properties:

$$P_R = P_R^\dagger, \quad P_L = P_L^\dagger, \quad P_R + P_L = \mathbb{1}, \quad P_R^2 = P_R, \quad P_L^2 = P_L, \quad P_R P_L = P_L P_R = 0.$$

3. Consider the extreme relativistic positive-energy solution with three-momentum $\vec{p}$,

$$u(\vec{p}, \pm) \approx \sqrt{E}\begin{pmatrix} \chi_\pm \\ \pm\chi_\pm \end{pmatrix} \equiv u_\pm(\vec{p}),$$

where we assume that the spin in the rest frame is either parallel or antiparallel to the direction of the momentum

$$\vec{\sigma} \cdot \hat{p} \chi_\pm = \pm \chi_\pm.$$

In the standard representation of Dirac matrices we find

$$P_R = \frac{1}{2} \begin{pmatrix} \mathbb{1}_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & \mathbb{1}_{2\times2} \end{pmatrix}, \quad P_L = \frac{1}{2} \begin{pmatrix} \mathbb{1}_{2\times2} & -\mathbb{1}_{2\times2} \\ -\mathbb{1}_{2\times2} & \mathbb{1}_{2\times2} \end{pmatrix}.$$

Show that

$$P_R u_+ = u_+, \quad P_L u_+ = 0, \quad P_R u_- = 0, \quad P_L u_- = u_-.$$

4. We have defined the left- and right-handed fields as

$$\begin{aligned} q_L &= P_L q, \quad q_R = P_R q, \\ \bar{q}_R &= q_R^\dagger \gamma_0 = q^\dagger P_R^\dagger \gamma_0 = q^\dagger P_R \gamma_0 = q^\dagger \gamma_0 P_L = \bar{q} P_L, \\ \bar{q}_L &= \bar{q} P_R. \end{aligned}$$

Show that

$$\bar{q}\Gamma_i q = \begin{cases} \bar{q}_R \Gamma_1 q_R + \bar{q}_L \Gamma_1 q_L & \text{for} \quad \Gamma_1 \in \{\gamma^\mu, \gamma^\mu \gamma_5\} \\ \bar{q}_R \Gamma_2 q_L + \bar{q}_L \Gamma_2 q_R & \text{for} \quad \Gamma_2 \in \{\mathbb{1}, \gamma_5, \sigma^{\mu\nu}\} \end{cases}.$$

Hint: Make use of $\{\Gamma_1, \gamma_5\} = 0$ and $[\Gamma_2, \gamma_5] = 0$ as well as the properties of the projection operators derived in Exercise 2.

5. Express the quark mass matrix

$$\mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

in terms of the λ matrices λ_0, λ_3, and λ_8.

6. Consider the Lagrangian

$$\mathcal{L}(\Phi_1, \Phi_2, \Phi_3) = \frac{1}{2} \partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2} \Phi_i \Phi_i - \frac{\lambda}{4} (\Phi_i \Phi_i)^2,$$

where $m^2 < 0$ and $\lambda > 0$. The Lagrangian has an internal O(3) symmetry. Determine the minimum of the potential

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2} \Phi_i \Phi_i + \frac{\lambda}{4} (\Phi_i \Phi_i)^2.$$

Express the Lagrangian in terms of the fields Φ_1, Φ_2, and η, where $\eta + v = \Phi_3$ and $v = \sqrt{-m^2/\lambda}$.

7. Consider the potential of Exercise 6 with an additional small term, explicitly breaking the symmetry,

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2} \Phi_i \Phi_i + \frac{\lambda}{4} (\Phi_i \Phi_i)^2 + a\Phi_3,$$

where $\lambda > 0$, $m^2 < 0$ and $a > 0$. Determine the new minimum $\langle \vec{\Phi} \rangle$ up to and including first order in a by using the ansatz $\langle \vec{\Phi} \rangle = \vec{\Phi}_0 + a\vec{\Phi}_1 + \mathcal{O}(a^2)$. Determine the corresponding extremal value of the potential to first order in a.

8. Consider the parameterization

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right), \quad \phi = \sum_{a=1}^{8} \lambda_a\phi_a \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

where the ϕ_a are Hermitian fields and $F_0 \approx 93$ MeV. Make use of the Gell-Mann matrices and express the physical fields in terms of the Cartesian components, e.g.,

$$\pi^+(x) = \frac{1}{\sqrt{2}}[\phi_1(x) - i\phi_2(x)].$$

9. Consider the Lagrangian

$$\mathcal{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{F_0^2}{4}\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right),$$

where

$$\chi = 2B_0 \underbrace{\begin{pmatrix} \hat{m} & 0 & 0 \\ 0 & \hat{m} & 0 \\ 0 & 0 & m_s \end{pmatrix}}_{\mathcal{M}}.$$

The matrix U is given by

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right), \quad \phi = \sum_{a=1}^{8} \lambda_a\phi_a \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}.$$

Expand the mass term to second order in the fields and determine the mass squares of the Goldstone bosons.

10. Under charge conjugation fields describing particles are mapped on fields describing antiparticles, i.e., $\pi^0 \mapsto \pi^0$, $\eta \mapsto \eta$, $\pi^+ \leftrightarrow \pi^-$, $K^+ \leftrightarrow K^-$, $K^0 \leftrightarrow \bar{K}^0$.

   (a) What does that mean for the matrix

$$\phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}?$$

   (b) Using $A^T B^T = (BA)^T$ show by induction $(A^T)^n = (A^n)^T$. In combination with (a) show that $U = \exp(i\phi/F_0) \overset{C}{\mapsto} U^T$.

   (c) Under charge conjugation the external fields transform as

$$v_\mu \mapsto -v_\mu^T, \quad a_\mu \mapsto a_\mu^T, \quad s \mapsto s^T, \quad p \mapsto p^T.$$

   Derive the transformation behavior of $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$, $\chi = 2B_0(s + ip)$, and $\chi^\dagger$.

   (d) Using (b) and (c) show that the covariant derivative of U under charge conjugation transforms as

$$D_\mu U \mapsto (D_\mu U)^T.$$

(e) Show that

$$\mathcal{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] + \frac{F_0^2}{4}\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)$$

is invariant under charge conjugation. Note that $(A^T)^\dagger = (A^\dagger)^T$ and $\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$.

(f) As an example, show the invariance of the L_3 term of $\mathcal{L}_4$ under charge conjugation:

$$L_3 \mathrm{Tr}\left[D_\mu U (D^\mu U)^\dagger D_\nu U (D^\nu U)^\dagger\right].$$

Hint: At the end you will need $(D_\mu U)^\dagger = -U^\dagger D_\mu U U^\dagger$ and $U^\dagger D_\mu U U^\dagger = -(D_\mu U)^\dagger$.

11. We will investigate the reaction $\gamma(q) + \pi^+(p) \to \gamma(q') + \pi^+(p')$ at lowest order in the momentum expansion $[\mathcal{O}(p^2)]$.

(a) Consider the first term of $\mathcal{L}_2$ and substitute

$$r_\mu = l_\mu = -eQ\mathcal{A}_\mu, \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \quad e > 0, \quad \frac{e^2}{4\pi} \approx \frac{1}{137},$$

where $\mathcal{A}_\mu$ is a Hermitian (external) electromagnetic field. Show that

$$\begin{aligned} D_\mu U &= \partial_\mu U + ie\mathcal{A}_\mu [Q, U], \\ (D^\mu U)^\dagger &= \partial^\mu U^\dagger + ie\mathcal{A}^\mu [Q, U^\dagger]. \end{aligned}$$

Using the substitution $U \leftrightarrow U^\dagger$, show that the resulting Lagrangian consists of terms involving only even numbers of Goldstone boson fields.

(b) Insert the result of (a) into $\mathcal{L}_2$ and verify

$$\begin{aligned} \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] &= \frac{F_0^2}{4}\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] \\ &\quad - ie\mathcal{A}_\mu \frac{F_0^2}{2}\mathrm{Tr}[Q(\partial^\mu U U^\dagger - U^\dagger \partial^\mu U)] \\ &\quad - e^2 \mathcal{A}_\mu \mathcal{A}^\mu \frac{F_0^2}{4}\mathrm{Tr}([Q, U][Q, U^\dagger]). \end{aligned}$$

Hint: $U\partial^\mu U^\dagger = -\partial^\mu U U^\dagger$ and $\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$.
The second term describes interactions with a single photon and the third term with two photons.

(c) Using $U = \exp(i\phi/F_0) = 1 + i\phi/F_0 - \phi^2/(2F_0^2) + \cdots$, identify those interaction terms which contain exactly two Goldstone bosons:

$$\begin{aligned} \mathcal{L}_2^{A-2\phi} &= -e\mathcal{A}_\mu \frac{i}{2}\mathrm{Tr}(Q[\partial^\mu \phi, \phi]), \\ \mathcal{L}_2^{2A-2\phi} &= -\frac{1}{4}e^2 \mathcal{A}_\mu \mathcal{A}^\mu \mathrm{Tr}([Q, \phi][Q, \phi]). \end{aligned}$$

(d) Insert ϕ of Exercise 8. Verify the intermediate steps

$$\begin{aligned} ([\partial^\mu \phi, \phi])_{11} &= 2(\partial^\mu \pi^+ \pi^- - \pi^+ \partial^\mu \pi^- + \partial^\mu K^+ K^- - K^+ \partial^\mu K^-), \\ ([\partial^\mu \phi, \phi])_{22} &= 2(\partial^\mu \pi^- \pi^+ - \pi^- \partial^\mu \pi^+ + \partial^\mu K^0 \bar{K}^0 - K^0 \partial^\mu \bar{K}^0), \\ ([\partial^\mu \phi, \phi])_{33} &= 2(\partial^\mu K^- K^+ - K^- \partial^\mu K^+ + \partial^\mu \bar{K}^0 K^0 - \bar{K}^0 \partial^\mu K^0), \end{aligned}$$

$$
\begin{aligned}
[Q,\phi] &= \sqrt{2}\begin{pmatrix} 0 & \pi^+ & K^+ \\ -\pi^- & 0 & 0 \\ -K^- & 0 & 0 \end{pmatrix}, \\
[Q,\phi][Q,\phi] &= -2\begin{pmatrix} \pi^+\pi^- + K^+K^- & 0 & 0 \\ 0 & \pi^-\pi^+ & \pi^-K^+ \\ 0 & K^-\pi^+ & K^-K^+ \end{pmatrix}.
\end{aligned}
$$

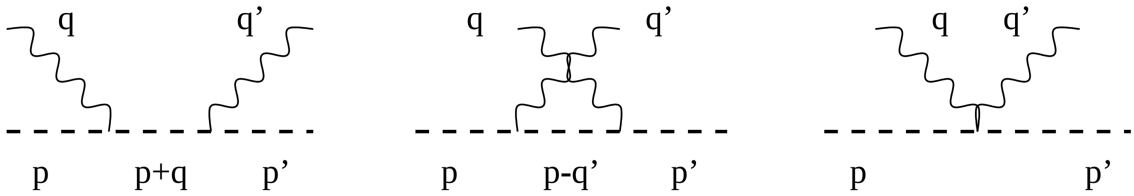
Now show

$$
\begin{aligned}
\mathcal{L}_2^{A-2\phi} &= -\mathcal{A}_\mu ie(\partial^\mu\pi^+\pi^- - \pi^+\partial^\mu\pi^- + \partial^\mu K^+K^- - K^+\partial^\mu K^-), \\
\mathcal{L}_2^{2A-2\phi} &= e^2\mathcal{A}_\mu\mathcal{A}^\mu(\pi^+\pi^- + K^+K^-).
\end{aligned}
$$

(e) The corresponding Feynman rules read

$$
\begin{aligned}
\mathcal{L}_2^{A-2\phi} \quad &\Rightarrow \quad \text{vertex for } \gamma(q,\epsilon)+\pi^\pm(p)\to\pi^\pm(p') \; : \; \mp ie\epsilon\cdot(p+p'), \\
\mathcal{L}_2^{2A-2\phi} \quad &\Rightarrow \quad \text{vertex for } \gamma(q,\epsilon)+\pi^\pm(p)\to\gamma(q',\epsilon')+\pi^\pm(p') \; : \; 2ie^2\epsilon'^*\cdot\epsilon,
\end{aligned}
$$

and analogously for charged kaons. An internal line of momentum p is described by the propagator $i/(p^2 - M^2 + i0^+)$. Determine the Compton scattering amplitude for $\gamma(q,\epsilon) + \pi^+(p) \to \gamma(q',\epsilon') + \pi^+(p')$:



What is the scattering amplitude for $\gamma(q,\epsilon) + \pi^-(p) \to \gamma(q',\epsilon') + \pi^-(p')$?

(f) Verify gauge invariance in terms of the substitution $q \to \epsilon$.

(g) Verify the invariance of the matrix element under the substitution $(q,\epsilon) \leftrightarrow (-q',\epsilon'^*)$ (photon crossing).

12. Consider the Lagrangian

$$
\mathcal{L}_2 = \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\right) + \frac{F^2}{4}\mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)
$$

in SU(2) with

$$
\chi = 2B\underbrace{\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}}_{M}
$$

and U given by

$$
U(x) = \exp\left(i\frac{\phi(x)}{F}\right), \quad \phi = \sum_{a=1}^{3}\tau_a\phi_a \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.
$$

(a) Show that $\mathcal{L}_2$ contains only even powers of ϕ,

$$
\mathcal{L}_2 = \mathcal{L}_2^{2\phi} + \mathcal{L}_2^{4\phi} + \cdots.
$$

(b) Since $\mathcal{L}_2$ does not produce a three-Goldstone-boson vertex, the scattering of two Goldstone bosons is described by a 4-Goldstone-boson contact interaction. Verify

$$\mathcal{L}_2^{4\phi} = \frac{1}{24F^2}\left[\mathrm{Tr}([\phi,\partial_\mu\phi]\phi\partial^\mu\phi) + B\mathrm{Tr}(M\phi^4)\right]$$

by using the expansion

$$U = 1 + i\frac{\phi}{F} - \frac{1}{2}\frac{\phi^2}{F^2} - \frac{i}{6}\frac{\phi^3}{F^3} + \frac{1}{24}\frac{\phi^4}{F^4} + \cdots.$$

Remark: An analogous formula would be obtained in SU(3) with the corresponding replacements.

(c) Show that the interaction Lagrangian can be written as

$$\mathcal{L}_2^{4\pi} = \frac{1}{6F^2}\left(\vec{\phi}\cdot\partial_\mu\vec{\phi}\vec{\phi}\cdot\partial^\mu\vec{\phi} - \vec{\phi}^2\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}\right) + \frac{M_\pi^2}{24F^2}(\vec{\phi}^2)^2,$$

where $M_\pi^2 = 2Bm$ at $\mathcal{O}(p^2)$.

(d) Derive the Feynman rule for $\pi^a(p_a) + \pi^b(p_b) \to \pi^c(p_c) + \pi^d(p_d)$:

$$\begin{aligned}\mathcal{M} &= i\left[\delta^{ab}\delta^{cd}\frac{s-M_\pi^2}{F^2} + \delta^{ac}\delta^{bd}\frac{t-M_\pi^2}{F^2} + \delta^{ad}\delta^{bc}\frac{u-M_\pi^2}{F^2}\right] \\ &\quad -\frac{i}{3F^2}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)(\Lambda_a + \Lambda_b + \Lambda_c + \Lambda_d),\end{aligned}$$

where we introduced $\Lambda_k = p_k^2 - M_\pi^2$.

(e) Using four-momentum conservation, show that the so-called Mandelstam variables $s = (p_a + p_b)^2$, $t = (p_a - p_c)^2$, and $u = (p_a - p_d)^2$ satisfy the relation

$$s + t + u = p_a^2 + p_b^2 + p_c^2 + p_d^2.$$

(f) The T-matrix element ($\mathcal{M} = iT$) of the scattering process $\pi^a(p_a)+\pi^b(p_b) \to \pi^c(p_c)+\pi^d(p_d)$ can be parameterized as

$$T^{ab;cd}(p_a, p_b; p_c, p_d) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,s,u) + \delta^{ad}\delta^{bc}A(u,t,s),$$

where the function A satisfies $A(s,t,u) = A(s,u,t)$. Since the pions form an isospin triplet, the two isovectors of both the initial and final states may be coupled to $I = 0, 1, 2$. For $m_u = m_d = m$ the strong interactions are invariant under isospin transformations, implying that scattering matrix elements can be decomposed as

$$\langle I', I_3'|T|I, I_3\rangle = T^I\delta_{II'}\delta_{I_3I_3'}.$$

For the case of $\pi\pi$ scattering the three isospin amplitudes are given in terms of the invariant amplitude A by

$$\begin{aligned}T^{I=0} &= 3A(s,t,u) + A(t,u,s) + A(u,s,t), \\ T^{I=1} &= A(t,u,s) - A(u,s,t), \\ T^{I=2} &= A(t,u,s) + A(u,s,t).\end{aligned}$$

For example, the physical $\pi^+\pi^+$ scattering process is described by $T^{I=2}$. Other physical processes are obtained using the appropriate Clebsch-Gordan coefficients. Evaluating the T matrices at threshold, one obtains the s-wave $\pi\pi$-scattering lengths

$$T^{I=0}|_{\mathrm{thr}} = 32\pi a_0^0, \quad T^{I=2}|_{\mathrm{thr}} = 32\pi a_0^2,$$

where the subscript 0 refers to s wave and the superscript to the isospin. ($T^{I=1}|_{\rm thr}$ vanishes because of Bose symmetry). Using the results of (d) verify the famous current-algebra prediction for the scattering lengths

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.156, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045,$$

where we replaced F by F_π and made use of the numerical values $F_\pi = 93.2$ MeV and $M_\pi = 139.57$ MeV.

Conclusion: Given that we know the value of F, the Lagrangian $\mathcal{L}_2$ *predicts* the low-energy scattering amplitude.

(g) You may repeat the above steps with a different parameterization of U which is very popular in SU(2) calculations:

$$U(x) = \frac{1}{F}[\sigma(x) + i\vec{\tau}\cdot\vec{\pi}(x)], \quad \sigma(x) = \sqrt{F^2 - \vec{\pi}^2(x)}.$$

Physical results do not depend on the parameterization. On the other hand, intermediate building blocks such as Feynman rules with off-mass-shell momenta depend on the parameterization chosen.

(h) You may also consider the SU(3) calculation which proceeds analogously. Using the properties of the Gell-Mann matrices show that

$$\begin{aligned}\mathcal{L}_2^{4\phi} &= -\frac{1}{6F_0^2}\phi_a\partial_\mu\phi_b\phi_c\partial^\mu\phi_d f_{abe}f_{cde} \\ &\quad + \frac{(2m+m_s)B_0}{36F_0^2}(\phi_a\phi_a)^2 + \frac{(m-m_s)B_0}{12\sqrt{3}F_0^2}\left(\frac{2}{3}\phi_8\phi_a\phi_b\phi_c d_{abc} + \phi_a\phi_a\phi_b\phi_c d_{bc8}\right).\end{aligned}$$

The following 5 exercises are related to dimensional regularization. If you are familiar with this issue you may proceed to Exercise 28.

13. Consider polar coordinates in 4 dimensions:

$$\begin{aligned} l_1 &= l\cos(\theta_1), \quad \theta_1 \in [0,\pi], \\ l_2 &= l\sin(\theta_1)\cos(\theta_2), \quad \theta_2 \in [0,\pi], \\ l_3 &= l\sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \quad \theta_3 \in [0,2\pi], \\ l_4 &= l\sin(\theta_1)\sin(\theta_2)\sin(\theta_3), \end{aligned}$$

where $l = \sqrt{l_1^2 + l_2^2 + l_3^2 + l_4^2}$. The transition from four-dimensional Cartesian coordinates to polar coordinates introduces the determinant of the Jacobi or functional matrix

$$J = \begin{pmatrix} \frac{\partial l_1}{\partial l} & \cdots & \frac{\partial l_1}{\partial \theta_3} \\ \vdots & & \vdots \\ \frac{\partial l_4}{\partial l} & \cdots & \frac{\partial l_4}{\partial \theta_3} \end{pmatrix}.$$

(a) Show that

$$\det(J) = l^3\sin^2(\theta_1)\sin(\theta_2)$$

and thus

$$dl_1 dl_2 dl_3 dl_4 = l^3 dl\, \underbrace{\sin^2(\theta_1)\sin(\theta_2)d\theta_1 d\theta_2 d\theta_3}_{d\Omega}.$$

(b) Verify

$$\int \mathrm{d}\Omega = 2\pi^2.$$

14. Show by induction

$$\int_0^\pi \sin^m(\theta)\mathrm{d}\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

for $m \geq 1$.

Hints: Make use of partial integration. $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$, $x\Gamma(x) = \Gamma(x+1)$.

15. We consider the integral

$$I = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+}.$$

Introduce $a \equiv \sqrt{\vec{k}^2 + M^2}$ and define

$$f(k_0) = \frac{1}{[k_0 + (a - i0^+)][k_0 - (a - i0^+)]}.$$

In order to determine $\int_{-\infty}^{\infty} \mathrm{d}k_0 f(k_0)$ as part of the calculation of I, we consider f in the complex k_0 plane and choose the paths

$$\gamma_1(t) = t, \quad a = -\infty, \quad b = +\infty \quad \text{and} \quad \gamma_2(t) = Re^{it}, \quad a = 0, \quad b = \pi.$$

(a) Using the residue theorem determine

$$\oint_C f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \lim_{R\to\infty} \int_{\gamma_2} f(z)\mathrm{d}z = 2\pi i \mathrm{Res}[f(z), -(a + i0^+)].$$

Verify

$$\int_{-\infty}^{\infty} \mathrm{d}k_0 f(k_0) = \frac{-i\pi}{\sqrt{\vec{k}^2 + M^2} - i0^+}.$$

(b) Using (a) show

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+} = \frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{\vec{k}^2 + M^2} - i0^+}.$$

(c) Now consider the generalization from $4 \to n$ dimensions:

$$\int \frac{\mathrm{d}^{n-1} k}{(2\pi)^{n-1}} \frac{1}{\sqrt{\vec{k}^2 + M^2}}, \qquad \vec{k}^2 = k_1^2 + k_2^2 + \cdots + k_{n-1}^2.$$

We can omit the $-i0^+$, because the integrand no longer has a pole. Introduce polar coordinates in $n-1$ dimensions and perform the angular integration to obtain

$$\int \frac{\mathrm{d}^{n-1} k}{(2\pi)^{n-1}} \frac{1}{\sqrt{\vec{k}^2 + M^2}} = \frac{1}{2^{n-2}} \pi^{-\frac{n-1}{2}} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \mathrm{d}r \frac{r^{n-2}}{\sqrt{r^2 + M^2}}.$$

(d) Using the substitutions $t = r/M$ and $y = t^2$ show

$$\int_0^\infty \mathrm{d}r \frac{r^{n-2}}{\sqrt{r^2 + M^2}} = \frac{1}{2} M^{n-2} \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(1 - \frac{n}{2}\right)}{\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}}.$$

Hint: Make use of the Beta function

$$B(x,y)=\int_0^\infty \frac{t^{x-1}dt}{(1+t)^{x+y}}=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

(e) Now put the results together to obtain

$$\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{i}{k^2-M^2+i0^+}=\frac{1}{(4\pi)^{\frac{n}{2}}}M^{n-2}\Gamma\left(1-\frac{n}{2}\right),$$

which agrees with the result of the lecture.

16. Show that in dimensional regularization

$$\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{(k^2)^p}{(k^2-M^2+i0^+)^q}=i(-)^{p-q}\frac{1}{(4\pi)^{\frac{n}{2}}}(M^2)^{p+\frac{n}{2}-q}\frac{\Gamma\left(p+\frac{n}{2}\right)\Gamma\left(q-p-\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(q)}.$$

We first assume $M^2>0$, $p=0,1,\cdots$, $q=1,2,\cdots$, and $p<q$. The last condition is used in the Wick rotation to guarantee that the quarter circles at infinity do not contribute to the integral.

(a) Show that the transition to the Euclidian metric produces the factor $i(-)^{p-q}$.

(b) Perform the angular integration in n dimensions.

(c) Perform the remaining radial integration using

$$\int_0^\infty \frac{l^{n-1}\mathrm{d}l}{(l^2+M^2)^\alpha}=\frac{1}{2}(M^2)^{\frac{n}{2}-\alpha}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)}.$$

What do you have to substitute for $n-1$ and α, respectively?

Now put the results together. The analytic continuation of the right-hand side is used to also define expressions with (integer) $q\le p$ in dimensional regularization.

17. Consider the complex function

$$f(z)=a^z=\exp(\ln(a)z)\equiv u(x,y)+iv(x,y),\quad a\in\mathbb{R}^+,\quad z=x+iy.$$

(a) Determine $u(x,y)$ and $v(x,y)$. Note that $u,v\in C^\infty(\mathbb{R}^2)$.

(b) Determine $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$. Show that the Cauchy-Riemann differential equations $\partial u/\partial x=\partial v/\partial y$ and $\partial u/\partial y=-\partial v/\partial x$ are satisfied. The complex function f is thus holomorphic in $\mathbb{C}$. We made use of this fact when discussing $I(M^2,\mu^2,n)$ as a function of the complex variable n in the context of dimensional regularization.

18. For the calculation of the Goldstone boson self energies at $\mathcal{O}(p^4)$ we need the interaction Lagrangian

$$\mathcal{L}_{\text{int}}=\mathcal{L}_2^{4\phi}+\mathcal{L}_4^{2\phi}.$$

Consider the Lagrangians of Gasser and Leutwyler and of Gasser, Sainio, and Švarc, respectively:

$$\begin{aligned}\mathcal{L}_4^{\text{GL}} &= \frac{l_1}{4}\left\{\text{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2+\frac{l_2}{4}\text{Tr}[D_\mu U(D_\nu U)^\dagger]\text{Tr}[D^\mu U(D^\nu U)^\dagger]\\ &\quad+\frac{l_3}{16}\left[\text{Tr}(\chi U^\dagger+U\chi^\dagger)\right]^2+\frac{l_4}{4}\text{Tr}[D_\mu U(D^\mu\chi)^\dagger+D_\mu\chi(D^\mu U)^\dagger]\end{aligned}$$

$$
\begin{aligned}
&+l_5\left[\mathrm{Tr}(f^R_{\mu\nu}Uf_L^{\mu\nu}U^\dagger)-\frac{1}{2}\mathrm{Tr}(f^L_{\mu\nu}f_L^{\mu\nu}+f^R_{\mu\nu}f_R^{\mu\nu})\right]\\
&+i\frac{l_6}{2}\mathrm{Tr}[f^R_{\mu\nu}D^\mu U(D^\nu U)^\dagger+f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U]\\
&-\frac{l_7}{16}\left[\mathrm{Tr}(\chi U^\dagger-U\chi^\dagger)\right]^2\\
&+\frac{h_1+h_3}{4}\mathrm{Tr}(\chi\chi^\dagger)+\frac{h_1-h_3}{16}\left\{\left[\mathrm{Tr}(\chi U^\dagger+U\chi^\dagger)\right]^2\right.\\
&\left.+\left[\mathrm{Tr}(\chi U^\dagger-U\chi^\dagger)\right]^2-2\mathrm{Tr}(\chi U^\dagger\chi U^\dagger+U\chi^\dagger U\chi^\dagger)\right\}\\
&-2h_2\mathrm{Tr}(f^L_{\mu\nu}f_L^{\mu\nu}+f^R_{\mu\nu}f_R^{\mu\nu}).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_4^{\mathrm{GSS}} ={}& \frac{l_1}{4}\left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2+\frac{l_2}{4}\mathrm{Tr}[D_\mu U(D_\nu U)^\dagger]\mathrm{Tr}[D^\mu U(D^\nu U)^\dagger]\\
&+\frac{l_3+l_4}{16}\left[\mathrm{Tr}(\chi U^\dagger+U\chi^\dagger)\right]^2+\frac{l_4}{8}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\mathrm{Tr}(\chi U^\dagger+U\chi^\dagger)\\
&+l_5\mathrm{Tr}(f^R_{\mu\nu}Uf_L^{\mu\nu}U^\dagger)+i\frac{l_6}{2}\mathrm{Tr}[f^R_{\mu\nu}D^\mu U(D^\nu U)^\dagger+f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U]\\
&-\frac{l_7}{16}\left[\mathrm{Tr}(\chi U^\dagger-U\chi^\dagger)\right]^2+\frac{h_1+h_3-l_4}{4}\mathrm{Tr}(\chi\chi^\dagger)\\
&+\frac{h_1-h_3-l_4}{16}\left\{\left[\mathrm{Tr}(\chi U^\dagger+U\chi^\dagger)\right]^2+\left[\mathrm{Tr}(\chi U^\dagger-U\chi^\dagger)\right]^2\right.\\
&\left.-2\mathrm{Tr}(\chi U^\dagger\chi U^\dagger+U\chi^\dagger U\chi^\dagger)\right\}-\frac{4h_2+l_5}{2}\mathrm{Tr}(f^L_{\mu\nu}f_L^{\mu\nu}+f^R_{\mu\nu}f_R^{\mu\nu}).
\end{aligned}
$$

Setting the external fields to zero and inserting $\chi=2Bm$, derive the terms involving two pion fields.

Remark: The bare and the renormalized low-energy constants l_i and l_i^r are related by

$$l_i=l_i^r+\gamma_i\frac{R}{32\pi^2},$$

where $R=2/(n-4)-[\ln(4\pi)+\Gamma'(1)+1]$ and

$$\gamma_1=\frac{1}{3},\quad \gamma_2=\frac{2}{3},\quad \gamma_3=-\frac{1}{2},\quad \gamma_4=2,\quad \gamma_5=-\frac{1}{6},\quad \gamma_6=-\frac{1}{3},\quad \gamma_7=0.$$

In the SU(2) sector one often uses the scale-independent parameters $\bar{l}_i$ which are defined by

$$l_i^r=\frac{\gamma_i}{32\pi^2}\left[\bar{l}_i+\ln\left(\frac{M^2}{\mu^2}\right)\right],\quad i=1,\cdots,6,$$

where $M^2=B(m_u+m_d)$. Since $\ln(1)=0$, the $\bar{l}_i$ are proportional to the renormalized coupling constant at the scale $\mu=M$.

19. Using isospin symmetry, at $\mathcal{O}(p^4)$ the pion self energy is of the form

$$\Sigma_{ba}(p^2)=\delta_{ab}(A+Bp^2).$$

The constants A and B receive a tree-level contribution from $\mathcal{L}_4$ and a one-loop contribution from $\mathcal{L}_2$ (see Fig. 1). Using the results of exercises 12, 15, and 18, derive the expressions of Table 1 for the self-energy coefficients.

Using

$$M^2_{\pi,4}=\frac{M^2_{\pi,2}+A}{1-B}=M^2_{\pi,2}(1+B)+A+\mathcal{O}(p^6),$$

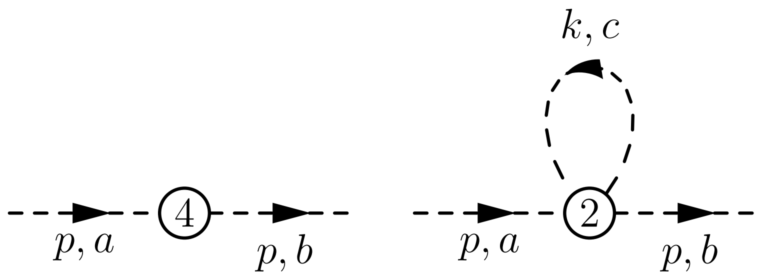


Figure 1: Self-energy diagrams at $\mathcal{O}(p^4)$. Vertices derived from $\mathcal{L}_{2n}$ are denoted by $2n$ in the interaction blobs.

Table 1: Self-energy coefficients and wave function renormalization constants. I denotes the dimensionally regularized integral $I = I(M^2, \mu^2, n) = \frac{M^2}{16\pi^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right] + O(n-4)$, $R = \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1]$, $M^2 = 2Bm$.

	A	B	Z
GL, exponential	$-\frac{1}{6}\frac{M^2}{F^2}I + 2l_3\frac{M^4}{F^2}$	$\frac{2}{3}\frac{I}{F^2}$	$1 + \frac{2}{3}\frac{I}{F^2}$
GL, square root	$\frac{3}{2}\frac{M^2}{F^2}I + 2l_3\frac{M^4}{F^2}$	$-\frac{I}{F^2}$	$1 - \frac{I}{F^2}$
GSS, exponential	$-\frac{1}{6}\frac{M^2}{F^2}I + 2(l_3 + l_4)\frac{M^4}{F^2}$	$\frac{2}{3}\frac{I}{F^2} - 2l_4\frac{M^2}{F^2}$	$1 + \frac{2}{3}\frac{I}{F^2} - 2l_4\frac{M^2}{F^2}$
GSS, square root	$\frac{3}{2}\frac{M^2}{F^2}I + 2(l_3 + l_4)\frac{M^4}{F^2}$	$-\frac{I}{F^2} - 2l_4\frac{M^2}{F^2}$	$1 - \frac{I}{F^2} - 2l_4\frac{M^2}{F^2}$

derive the squared pion mass at $\mathcal{O}(p^4)$:

$$M_{\pi,4}^2 = M^2 - \frac{\bar{l}_3}{32\pi^2 F^2}M^4 + \mathcal{O}(M^6),$$

where $M^2 = 2Bm$.

20. You may repeat the calculations in SU(3) to obtain the masses of the Goldstone boson octet.

    Remark: Conceptionally the calculation is completetly analogous to the SU(2) calculation. Due to the SU(3) algebra and the fact that the loop integrals contain different mass scales it is now considerably more work.

21. Using

$$B = \sum_{a=1}^{8} \frac{\lambda_a B_a}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix},$$

express the physical fields in terms of Cartesian fields.

22. Consider the lowest-order πN Lagrangian

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\not{D} - \overset{\circ}{m}_N + \frac{\overset{\circ}{g}_A}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi.$$

Assume that there are no external fields, $l_\mu = r_\mu = v_\mu^{(s)} = 0$, so that

$$\begin{aligned}\Gamma_\mu &= \frac{1}{2}(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger),\\ u_\mu &= i(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger).\end{aligned}$$

By expanding

$$u = \exp\left(i\frac{\vec{\tau}\cdot\vec{\phi}}{2F}\right) = 1 + i\frac{\vec{\tau}\cdot\vec{\phi}}{2F} - \frac{\vec{\phi}^2}{8F^2} + \cdots,$$

derive the interaction Lagrangians containing one and two pion fields, respectively.

23. Consider the SU(2)-valued function

$$K(L, R, U) = \sqrt{RUL^\dagger}^{-1} R\sqrt{U}.$$

Verify the homomorphism property

$$K(L_1, R_1, R_2UL_2^\dagger)K(L_2, R_2, U) = K((L_1L_2), (R_1R_2), U).$$

24. Consider

$$u_\mu \equiv i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right].$$

Using

$$u' = V_R u K^\dagger = K u V_L^\dagger$$

show that, under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$, u_μ transforms as

$$u_\mu \mapsto K u_\mu K^\dagger.$$

25. Consider the two-flavor Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_2^\pi,$$

where

$$\begin{aligned}\mathcal{L}_{\pi N}^{(1)} &= \bar{\Psi}\left(i\not{D} - \overset{\circ}{m}_N + \frac{\overset{\circ}{g}_A}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi,\\ \mathcal{L}_2^\pi &= \frac{F^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger] + \frac{F^2}{4}\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger).\end{aligned}$$

(a) We would like to study this Lagrangian in the presence of an electromagnetic field $\mathcal{A}_\mu$. For that purpose we need to insert for the external fields

$$r_\mu = l_\mu = -e\frac{\tau_3}{2}\mathcal{A}_\mu, \quad v_\mu^{(s)} = -\frac{e}{2}\mathcal{A}_\mu.$$

Derive the interaction Lagrangians $\mathcal{L}_{\gamma NN}$, $\mathcal{L}_{\pi NN}$, $\mathcal{L}_{\gamma\pi NN}$, and $\mathcal{L}_{\gamma\pi\pi}$. Here, the nomenclature is such that $\mathcal{L}_{\gamma NN}$ denotes the interaction Lagrangian describing the interaction of an external electromagnetic field with a single nucleon in the initial and final states, respectively. For example, $\mathcal{L}_{\gamma\pi NN}$ must be symbolically of the type $\bar{\Psi}\phi\mathcal{A}\Psi$. Using Feynman rules, these four interaction Lagrangians would be sufficient to describe pion photoproduction of the nucleon, $\gamma N \to \pi N$, at lowest order in ChPT.

(b) Now we would like to describe the interaction with a massive charged weak boson $\mathcal{W}^{\pm}_\mu = (\mathcal{W}_{1\mu} \mp i\mathcal{W}_{2\mu})/\sqrt{2}$,

$$r_\mu = 0, \quad l_\mu = -\frac{g}{\sqrt{2}}(\mathcal{W}^+_\mu T_+ + h.c.),$$

where $h.c.$ refers to the Hermitian conjugate and

$$T_+ = \begin{pmatrix} 0 & V_{ud} \\ 0 & 0 \end{pmatrix}.$$

Here, V_{ud} denotes an element of the Cabibbo-Kobayashi-Maskawa quark-mixing matrix,

$$|V_{ud}| = 0.97425 \pm 0.00022.$$

At lowest order in perturbation theory, the Fermi constant is related to the gauge coupling g and the W mass as

$$G_F = \sqrt{2}\frac{g^2}{8M_W^2} = 1.16637(1) \times 10^{-5}\,\mathrm{GeV}^{-2}.$$

Derive the interaction Lagrangians $\mathcal{L}_{WNN}$ and $\mathcal{L}_{W\pi}$.

(c) Finally, we consider the neutral weak interaction

$$\begin{aligned} r_\mu &= e\tan(\theta_W)\frac{\tau_3}{2}\mathcal{Z}_\mu, \\ l_\mu &= -\frac{g}{\cos(\theta_W)}\frac{\tau_3}{2}\mathcal{Z}_\mu + e\tan(\theta_W)\frac{\tau_3}{2}\mathcal{Z}_\mu, \\ v^{(s)}_\mu &= \frac{e\tan(\theta_W)}{2}\mathcal{Z}_\mu, \end{aligned}$$

where θ_W is the weak angle, $e = g\sin(\theta_W)$. Derive the interaction Lagrangians $\mathcal{L}_{ZNN}$ and $\mathcal{L}_{Z\pi}$.

26. Consider the three-flavor Lagrangian

$$\mathcal{L}^{(1)}_{MB} = \mathrm{Tr}\left[\bar{B}\left(i\not{D} - M_0\right)B\right] - \frac{D}{2}\mathrm{Tr}\left(\bar{B}\gamma^\mu\gamma_5\{u_\mu, B\}\right) - \frac{F}{2}\mathrm{Tr}\left(\bar{B}\gamma^\mu\gamma_5[u_\mu, B]\right)$$

in the absence of external fields:

$$\begin{aligned} D_\mu B &= \partial_\mu B + \frac{1}{2}[u^\dagger\partial_\mu u + u\partial_\mu u^\dagger, B], \\ u_\mu &= i(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger). \end{aligned}$$

Using

$$B = \frac{B_a\lambda_a}{\sqrt{2}}, \quad \bar{B} = \frac{\bar{B}_b\lambda_b}{\sqrt{2}},$$

show that the interaction Lagrangians with one and two mesons can be written as

$$\begin{aligned} \mathcal{L}^{(1)}_{\phi BB} &= \frac{1}{F_0}(d_{abc}D + if_{abc}F)\bar{B}_b\gamma^\mu\gamma_5 B_a\partial_\mu\phi_c, \\ \mathcal{L}^{(1)}_{\phi\phi BB} &= -\frac{i}{2F_0^2}f_{abe}f_{cde}\bar{B}_b\gamma^\mu B_a\phi_c\partial_\mu\phi_d. \end{aligned}$$

Hint: $u^\dagger\partial_\mu u + u\partial_\mu u^\dagger = u^\dagger\partial_\mu u - \partial_\mu u u^\dagger = [u^\dagger, \partial_\mu u]$.

27. Consider the general parameterization of the invariant amplitude $\mathcal{M} = iT$ for the process $\pi^a(q) + N(p) \to \pi^b(q') + N(p')$:

$$
\begin{aligned}
T^{ab}(p,q;p',q') &= \bar{u}(p')\Bigg\{ \underbrace{\frac{1}{2}\{\tau^b,\tau^a\}}_{\delta^{ab}} A^+(\nu,\nu_B) + \underbrace{\frac{1}{2}[\tau^b,\tau^a]}_{-i\epsilon_{abc}\tau^c} A^-(\nu,\nu_B) \\
&\quad + \frac{1}{2}(\not{q}+\not{q}')\left[\delta^{ab}B^+(\nu,\nu_B) - i\epsilon_{abc}\tau^c B^-(\nu,\nu_B)\right]\Bigg\} u(p),
\end{aligned}
$$

with the two independent scalar kinematical variables

$$
\begin{aligned}
\nu &= \frac{s-u}{4m_N} = \frac{(p+p')\cdot q}{2m_N} = \frac{(p+p')\cdot q'}{2m_N}, \\
\nu_B &= -\frac{q\cdot q'}{2m_N} = \frac{t-2M_\pi^2}{4m_N},
\end{aligned}
$$

where $s=(p+q)^2$, $t=(p'-p)^2$, and $u=(p'-q)^2$ are the usual Mandelstam variables satisfying $s+t+u = 2m_N^2 + 2M_\pi^2$.

(a) Show that

$$
s - m_N^2 = 2m_N(\nu-\nu_B), \quad u - m_N^2 = -2m_N(\nu+\nu_B).
$$

Hint: Make use of four-momentum conservation, $p+q=p'+q'$, and of the mass-shell conditions, $p^2=p'^2=m_N^2$, $q^2=q'^2=M_\pi^2$.

Derive the threshold values

$$
\nu|_{\mathrm{thr}} = M_\pi, \quad \nu_B|_{\mathrm{thr}} = -\frac{M_\pi^2}{2m_N}.
$$

(b) Show that from pion-crossing symmetry

$$
T^{ab}(p,q;p',q') = T^{ba}(p,-q';p',-q)
$$

we obtain for the crossing behavior of the amplitudes

$$
\begin{aligned}
A^+(-\nu,\nu_B) &= A^+(\nu,\nu_B), \quad A^-(-\nu,\nu_B) = -A^-(\nu,\nu_B), \\
B^+(-\nu,\nu_B) &= -B^+(\nu,\nu_B), \quad B^-(-\nu,\nu_B) = B^-(\nu,\nu_B).
\end{aligned}
$$

(c) The physical πN channels may be expressed in terms of the isospin eigenstates as

$$
\begin{aligned}
|p\pi^+\rangle &= |\frac{3}{2},\frac{3}{2}\rangle, \\
|p\pi^0\rangle &= \sqrt{\frac{2}{3}}|\frac{3}{2},\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}|\frac{1}{2},\frac{1}{2}\rangle, \\
|n\pi^+\rangle &= \frac{1}{\sqrt{3}}|\frac{3}{2},\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|\frac{1}{2},\frac{1}{2}\rangle, \\
|p\pi^-\rangle &= \frac{1}{\sqrt{3}}|\frac{3}{2},-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|\frac{1}{2},-\frac{1}{2}\rangle, \\
|n\pi^0\rangle &= \sqrt{\frac{2}{3}}|\frac{3}{2},-\frac{1}{2}\rangle - \frac{1}{\sqrt{3}}|\frac{1}{2},-\frac{1}{2}\rangle, \\
|n\pi^-\rangle &= |\frac{3}{2},-\frac{3}{2}\rangle.
\end{aligned}
$$

Using

$$\langle I', I'_3|T|I, I_3\rangle = T^I \delta_{II'}\delta_{I_3 I'_3},$$

derive the expressions for $\langle p\pi^0|T|n\pi^+\rangle$, $\langle p\pi^0|T|p\pi^0\rangle$, and $\langle n\pi^+|T|n\pi^+\rangle$. Verify that

$$\langle p\pi^0|T|p\pi^0\rangle - \langle n\pi^+|T|n\pi^+\rangle = \frac{1}{\sqrt{2}}\langle p\pi^0|T|n\pi^+\rangle.$$

(d) Consider the so-called pseudoscalar pion-nucleon interaction

$$\mathcal{L}^{\mathrm{PS}}_{\pi NN} = -ig_{\pi N}\bar{\Psi}\gamma_5\vec{\tau}\cdot\vec{\phi}\Psi.$$

The Feynman rule for both the absorption and the emission of a pion with Cartesian isospin index a is given by

$$g_{\pi N}\gamma_5\tau_a.$$

Derive the invariant amplitude for the s- and u-channel contributions.

28. In the following we will calculate the mass m_N of the nucleon up to and including order $\mathcal{O}(q^3)$. As in the case of pions, the physical mass is defined through the pole of the full propagator (at $\not{p} = m_N$ for the nucleon). The (unrenormalized) propagator is given by

$$S_0(p) = \frac{1}{\not{p} - m_0 - \Sigma_0(\not{p})} \equiv \frac{1}{\not{p} - m - \Sigma(\not{p})}, \tag{1}$$

where m_0 refers to the bare mass, m is the nucleon mass in the chiral limit, and $\Sigma_0(\not{p})$ denotes the nucleon self energy as a function of $\not{p}$ (note that $\not{p}\not{p} = p^2$). To determine the mass, the equation

$$m_N - m_0 - \Sigma_0(m_N) = m_N - m - \Sigma(m_N) = 0 \tag{2}$$

has to be solved, so the task is to calculate the nucleon self energy $\Sigma(\not{p})$.

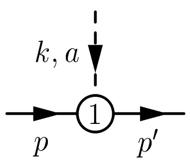
(a) The πN Lagrangian at order $\mathcal{O}(q^2)$ is given by

$$\begin{aligned}\mathcal{L}^{(2)}_{\pi N} &= c_1\mathrm{Tr}(\chi_+)\bar{\Psi}\Psi - \frac{c_2}{4m^2}\mathrm{Tr}(u_\mu u_\nu)(\bar{\Psi}D^\mu D^\nu\Psi + \text{H. c.})\\ &\quad + \frac{c_3}{2}\mathrm{Tr}(u^\mu u_\mu)\bar{\Psi}\Psi - \frac{c_4}{4}\bar{\Psi}\gamma^\mu\gamma^\nu[u_\mu, u_\nu]\Psi + c_5\bar{\Psi}\left[\chi_+ - \frac{1}{2}\mathrm{Tr}(\chi_+)\right]\Psi\\ &\quad + \bar{\Psi}\left[\frac{c_6}{2}f^+_{\mu\nu} + \frac{c_7}{2}v^{(s)}_{\mu\nu}\right]\sigma^{\mu\nu}\Psi.\end{aligned}$$

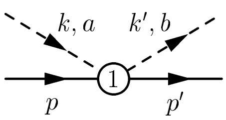
Which of these terms contain only the nucleon fields and therefore give a contact contribution to the self energy? Determine $-i\Sigma^{\mathrm{contact}}(\not{p})$ from $i\langle\bar{\Psi}|\mathcal{L}^{(2)}_{\pi N}|\Psi\rangle$.

Remark: There are no contact contributions from the Lagrangian $\mathcal{L}^{(3)}_{\pi N}$.

(b) By using the expansion of $\mathcal{L}^{(1)}_{\pi N}$ up to two pion fields from Assignment 12 verify the following Feynman rules:[1]



$$-\frac{g_{A0}}{2F_0}\not{k}\gamma_5\tau_a$$



$$\frac{1}{4F_0^2}(\not{k} + \not{k}')\epsilon_{abi}\tau_i$$

[1]Here, the subscripts 0 denote bare quantities.

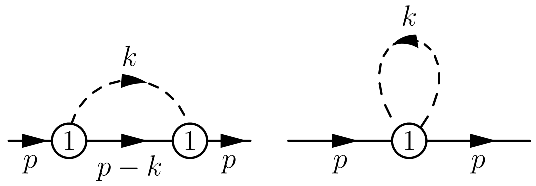


Figure 2: One-loop contributions to the nucleon self energy

There are two types of loop contributions at order $\mathcal{O}(q^3)$, shown in Figure 2.

(c) Use the Feynman rules to show that the second diagram in Figure 2 does not contribute to the self energy.

(d) Use the Feynman rules and the expressions for the propagators,

$$\begin{aligned} i\Delta_\pi(p) &= \frac{i}{p^2 - M^2 + i0^+}, \\ iS_N(p) &= i\frac{\not p + m}{p^2 - m^2 + i0^+}, \end{aligned}$$

to verify that in dimensional regularization the first diagram in Figure 2 gives the contribution

$$-i\Sigma^{\mathrm{loop}}(\not p) = -i\frac{3\mathrm{g}_{A0}^2}{4F_0^2}\, i\mu^{4-n}\int \frac{d^n k}{(2\pi)^n}\frac{\not k(\not p - m - \not k)\not k}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}. \tag{3}$$

(e) Show that the numerator can be simplified to

$$-(\not p + m)k^2 + (p^2 - m^2)\not k - \left[(p-k)^2 - m^2\right]\not k, \tag{4}$$

which, when inserted in Eq. (3), gives

$$\begin{aligned} \Sigma^{\mathrm{loop}}(\not p) = \frac{3\mathrm{g}_{A0}^2}{4F_0^2}&\left\{-(\not p + m)\mu^{4-n} i\int \frac{d^n k}{(2\pi)^n}\frac{1}{[(p-k)^2 - m^2 + i0^+]}\right. \\ &-(\not p + m)M^2\mu^{4-n} i\int \frac{d^n k}{(2\pi)^n}\frac{1}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} \\ &+(p^2 - m^2)\mu^{4-n} i\int \frac{d^n k}{(2\pi)^n}\frac{\not k}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} \\ &\left.-\mu^{4-n} i\int \frac{d^n k}{(2\pi)^n}\frac{\not k}{[k^2 - M^2 + i0^+]}\right\}. \end{aligned} \tag{5}$$

Hint: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $\{\gamma_\mu, \gamma_5\} = 0$, $\gamma_5\gamma_5 = 1$, $k^2 = k^2 - M^2 + M^2$.

(f) The last term in Eq. (5) vanishes since the integrand is odd in k. We use the following convention for scalar loop integrals

$$\begin{aligned} &I_{N\cdots\pi\cdots}(p_1, \cdots, q_1, \cdots) \\ &= \mu^{4-n} i\int \frac{d^n k}{(2\pi)^n}\frac{1}{[(k+p_1)^2 - m^2 + i0^+]\cdots[(k+q_1)^2 - M^2 + i0^+]\cdots}. \end{aligned}$$

To determine the vector integral use the ansatz

$$\mu^{4-n} i\int \frac{d^n k}{(2\pi)^n}\frac{k_\mu}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} = p_\mu C. \tag{6}$$

Multiply Eq. (6) by p^μ to show that C is given by

$$C = \frac{1}{2p^2}\left[I_N - I_\pi + (p^2 - m^2 + M^2)I_{N\pi}(-p,0)\right]. \tag{7}$$

Using the above convention the loop contribution to the nucleon self energy reads

$$\begin{aligned}\Sigma^{\text{loop}}(\not p) &= -\frac{3\mathrm{g}_{A0}^2}{4F_0^2}\left\{(\not p + m)I_N + (\not p + m)M^2 I_{N\pi}(-p,0)\right.\\ &\left. -(p^2-m^2)\frac{\not p}{2p^2}\left[I_N - I_\pi + (p^2 - m^2 + M^2)I_{N\pi}(-p,0)\right]\right\}.\end{aligned} \tag{8}$$

The explicit expressions for the integrals are given by

$$\begin{aligned}I_\pi &= \frac{M^2}{16\pi^2}\left[R + \ln\left(\frac{M^2}{\mu^2}\right)\right],\\ I_N &= \frac{m^2}{16\pi^2}\left[R + \ln\left(\frac{m^2}{\mu^2}\right)\right],\\ I_{N\pi}(p,0) &= \frac{1}{16\pi^2}\left[R + \ln\left(\frac{m^2}{\mu^2}\right) - 1\right.\\ &\left. + \frac{p^2 - m^2 + M^2}{p^2}\ln\left(\frac{M}{m}\right) + \frac{2mM}{p^2}F(\Omega)\right],\end{aligned} \tag{9}$$

where

$$\begin{aligned}R &= \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1],\\ \Omega &= \frac{p^2 - m^2 - M^2}{2mM},\end{aligned}$$

and

$$F(\Omega) = \begin{cases}\sqrt{\Omega^2 - 1}\ln\left(-\Omega - \sqrt{\Omega^2-1}\right), & \Omega \le -1,\\ \sqrt{1-\Omega^2}\ \arccos(-\Omega), & -1 \le \Omega \le 1,\\ \sqrt{\Omega^2-1}\ln\left(\Omega + \sqrt{\Omega^2-1}\right) - i\pi\sqrt{\Omega^2-1}, & 1 \le \Omega.\end{cases}$$

(g) The result for the self energy contains divergences as $n \to 4$ (the terms proportional to R), so it has to be renormalized. For convenience, choose the renormalization parameter $\mu = m$. The $\widetilde{\text{MS}}$ renormalization can be performed by simply dropping the terms proportional to R and by replacing all bare coupling constants ($c_{10}, \mathrm{g}_{A0}, F_0$) with the renormalized ones, now indicated by a subscript r. The $\widetilde{\text{MS}}$ renormalized self energy contribution then reads

$$\begin{aligned}\Sigma_r^{\text{loop}}(\not p) &= -\frac{3\mathrm{g}_{Ar}^2}{4F_r^2}\left\{(\not p + m)M^2 I_{N\pi}^r(-p,0)\right.\\ &\left. -(p^2-m^2)\frac{\not p}{2p^2}\left[(p^2 - m^2 + M^2)I_{N\pi}^r(-p,0) - I_\pi^r\right]\right\},\end{aligned} \tag{10}$$

where the superscript r on the integrals means that the terms proportional to R have been dropped. Using the definition of the integrals, show that Eq. (10) contains a term of order $\mathcal{O}(q^2)$. What does the presence of this term tell you about the applicability of the $\widetilde{\text{MS}}$ scheme in baryon ChPT?

Hint: What chiral order did the power counting assign to the diagram from which we calculated Σ^{loop}?

(h) We can now solve Eq. (2) for the nucleon mass,

$$\begin{aligned} m_N &= m + \Sigma_r^{\text{contact}}(m_N) + \Sigma_r^{\text{loop}}(m_N) \\ &= m - 4c_{1r}M^2 + \Sigma_r^{\text{loop}}(m_N). \end{aligned} \tag{11}$$

We have $m_N - m = \mathcal{O}(q^2)$. Since our calculation is only valid up to order $\mathcal{O}(q^3)$, determine $\Sigma_r^{\text{loop}}(m_N)$ to that order. Check that you only need an expansion of $I_{N\pi}^r$, which, using

$$\arccos(-\Omega) = \frac{\pi}{2} + \cdots,$$

verify to be

$$I_{N\pi}^r = \frac{1}{16\pi^2}\left(-1 + \frac{\pi M}{m} + \cdots\right). \tag{12}$$

Show that this yields

$$m_N = m - 4c_{1r}M^2 + \frac{3\mathrm{g}_{Ar}^2 M^2}{32\pi^2 F_r^2} m - \frac{3\mathrm{g}_{Ar}^2 M^3}{32\pi F_r^2}. \tag{13}$$

(i) The solution to the power counting problem is the observation that the term violating the power counting (the third on the right of Eq. (13)) is *analytic* in small quantities and can thus be absorbed in counter terms. In addition to the $\widetilde{\text{MS}}$ scheme we have to perform an additional *finite* renormalization. Rewrite

$$c_{1r} = c_1 + \delta c_1 \tag{14}$$

in Eq. (13) and determine δc_1 so that the term violating the power counting is absorbed, which then gives the final result for the nucleon mass at order $\mathcal{O}(q^3)$

$$m_N = m - 4c_1 M^2 - \frac{3\mathrm{g}_A^2 M^3}{32\pi F^2}. \tag{15}$$

All quantities appearing in Eq. (15) are now in the extended on-mass-shell (EOMS) scheme. For a detailed discussion of renormalization and power counting in manifestly Lorentz-invariant baryon chiral perturbation theory see T. Fuchs, J. Gegelia, G. Japaridze, and S. Scherer, Phys. Rev. D **68**, 056005 (2003).