



THE LOW-ENERGY FRONTIER  
OF THE STANDARD MODEL



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## Spontaneous Symmetry Breaking in QCD and Chiral Dynamics

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## **0. Motivation**

### **1. QCD and Chiral Symmetry**

### **2. Spontaneous Symmetry Breaking**

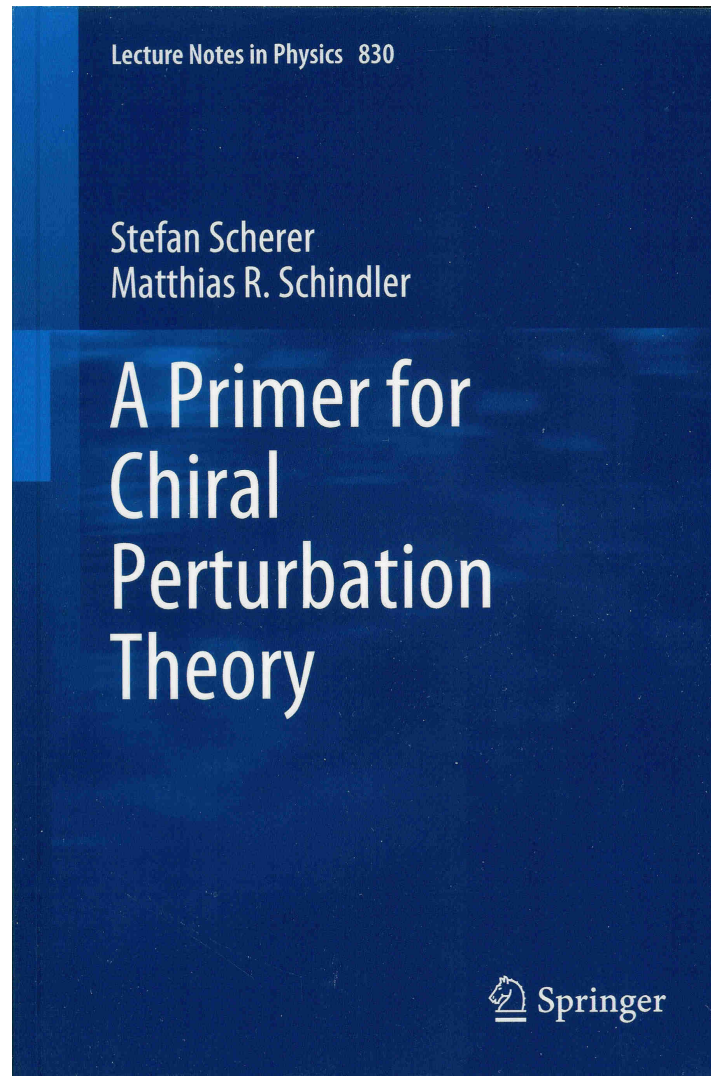
### **3. Chiral Perturbation Theory for Mesons**

### **4. Chiral Perturbation Theory for Baryons**

### **5. Chiral Effective Field Theory (Including Resonances, Constraints, ...)**

**Exercises available**

Material based on



S. Scherer und M. R. Schindler,  
**A Primer for Chiral Perturbation Theory**  
(Lecture Notes in Physics 830,  
Springer, Berlin, Heidelberg, 2012)

## Motivation of Keywords

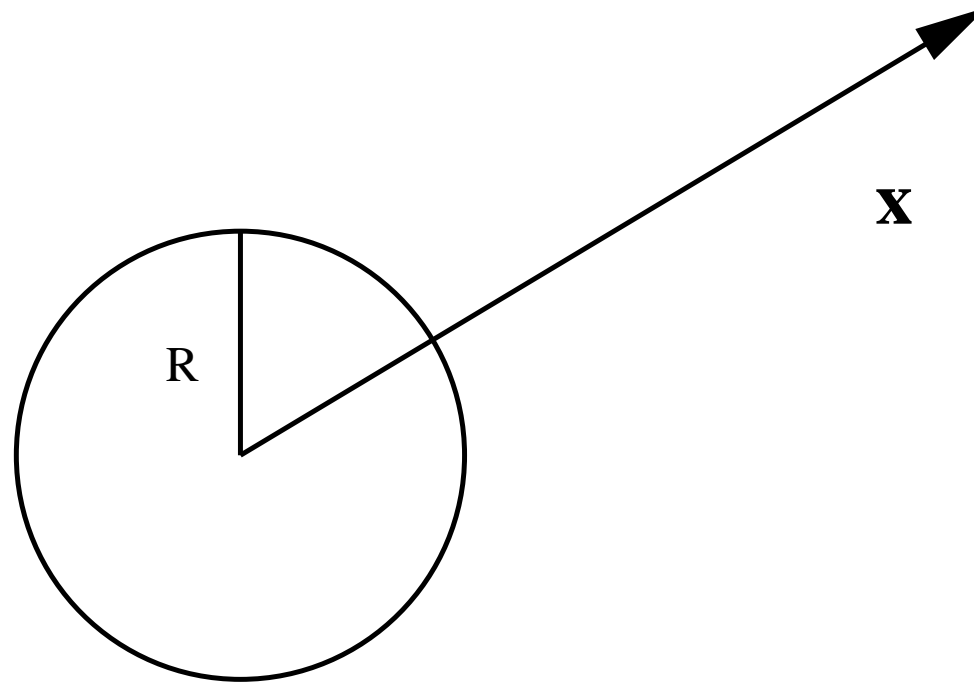
- Chiral perturbation theory (**ChPT**) is the effective field theory (**EFT**) of the Standard Model/strong interactions at low energies.
- EFTs are **low-energy approximations** to (more) fundamental theories.
- Instead of solving the underlying theory, low-energy physics is described with a set of variables (**effective degrees of freedom**) that is suited for the particular energy region you are interested in.
- In our case: **Pions and nucleons** instead of the more fundamental quarks and gluons of QCD.

- Calculate physical quantities in terms of an **expansion in  $p/\Lambda$** , where  $p$  stands for momenta or masses that are smaller than a certain momentum scale  $\Lambda$ .
- There exists a regime where both fundamental and effective theories yield the same results.
- EFTs are based on the **most general Lagrangian**, which includes all terms that are compatible with the symmetries of the underlying theory.  $\Rightarrow$  **Infinite number of terms**. Each term is accompanied by a low-energy coupling constant (**LEC**).
- One needs a method that allows one to decide which terms contribute in a calculation up to a certain accuracy: **Weinberg's power counting**.

- In actual calculations only a finite number of terms in the expansion in  $p/\Lambda$  has to be considered.  $\Rightarrow$  **Predictive power.**
- Effective field theories are non-renormalizable in the traditional sense. However, as long as one considers **all terms that are allowed by the symmetries**, divergences that occur in calculations up to any given order of  $p/\Lambda$  can be renormalized by redefining fields and parameters of the Lagrangian of the effective field theory. **The so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories.**

- First example from electrostatics illustrating the idea of a scale (here distance scale).

Consider charge distribution  $\rho(\vec{x}')$  which is localized inside a sphere of radius  $R$ .



Potential from solution to Poisson equation,

$$\Delta\phi = -\rho,$$

reads

$$\phi(\vec{x}) = \frac{1}{4\pi} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'.$$

Make use of

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi).$$

$\Rightarrow$

- Solution for  $|\vec{x}| \lesssim R$  **complicated**
- Solution for  $|\vec{x}| \gg R$  **simple**, because

$$\phi(\vec{x}) = \sum_{l,m} \underbrace{\left[ \int Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{x}') d^3x' \right]}_{\text{multipole moment } q_{lm}} \frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}.$$

- Far away, only the leading-order terms contribute.
- Systematic improvement possible.



- For smaller  $r$ , higher multipoles become more important.
- $q_{lm}$  parameterize short-distance physics.
- $\frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$  determine the long-distance effects of short-distance physics.

- (Simplified) analogies<sup>1</sup>

Multipole expansion	EFT
$q_{lm}$	LECs
$\frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$	Structures of most general $\mathcal{L}_{\text{EFT}}$

- Here: Simple separation of scales ( $R$ ).
- ChPT: Scales depend on underlying dynamics and masses of the participating particles.

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<sup>1</sup>Observables will be calculated in perturbation theory using  $\mathcal{L}_{\text{EFT}}$ .

- **Second example: “Integrating out” a heavy degree of freedom in a toy model ( $m \ll M$ ):**

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - M^2 \Phi^2) + \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) - \frac{\lambda}{2} \Phi \varphi^2.$$

**Equations of motion:**

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} - \frac{\partial \mathcal{L}}{\partial \Phi} = \square \Phi + M^2 \Phi + \frac{\lambda}{2} \varphi^2 = 0, \quad (*)$$

$$\square \varphi + m^2 \varphi + \lambda \varphi \Phi = 0. \quad (**)$$

**Formally solve (\*):**

$$\Phi = -\frac{\lambda}{2M^2} \frac{1}{1 + \frac{\square}{M^2}} \varphi^2.$$

**Insert solution into (\*\*).  $\Rightarrow$**

$$\square \varphi + m^2 \varphi - \frac{\lambda^2}{2M^2} \varphi \frac{1}{1 + \frac{\square}{M^2}} \varphi^2 = 0$$

and expand to leading order in  $1/M^2$ :

$$\square\varphi + m^2\varphi - \frac{\lambda^2}{2M^2}\varphi^3 = 0.$$

What is the corresponding **effective** Lagrangian to this order?

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2) + \frac{\lambda^2}{8M^2}\varphi^4.$$

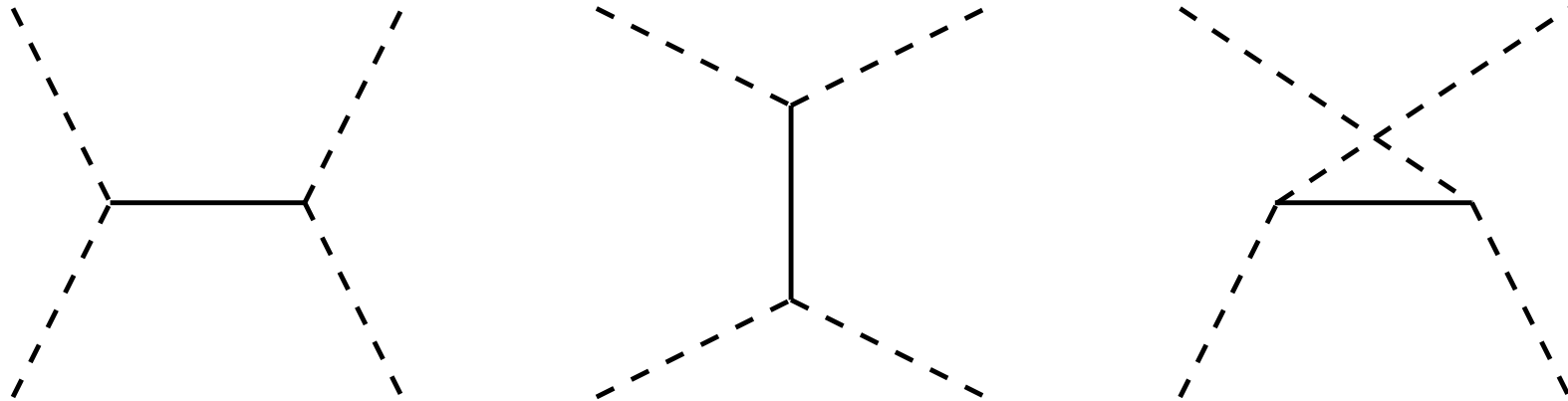
Compare with original Lagrangian:

- Heavy degree of freedom is gone.
- A different interaction term has appeared.

Do the two Lagrangians produce the same low-energy scattering amplitude for  $\varphi(p_1) + \varphi(p_2) \rightarrow \varphi(p_3) + \varphi(p_4)$ ?

Calculation with original Lagrangian.

Dotted line: Light particle; solid line: Heavy particle.



Mandelstam variables

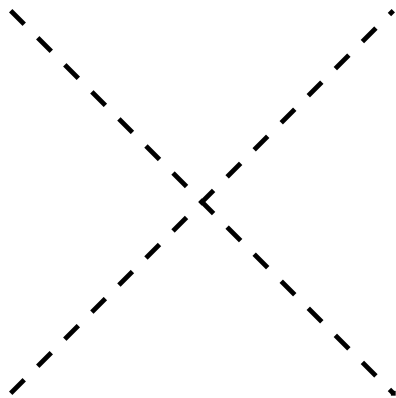
$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\ t &= (p_1 - p_3)^2 = (p_4 - p_2)^2, \\ u &= (p_1 - p_4)^2 = (p_3 - p_2)^2, \\ s + t + u &= 4m^2. \end{aligned}$$

**Condition:**  $\{s, |t|, |u|\} \ll M^2 = \Lambda^2$ . (\*)

**Result:**

$$\begin{aligned} \mathcal{M}_{\text{fund}} &= (-i\lambda)^2 \left( \frac{i}{s - M^2 + i0^+} + \frac{i}{t - M^2 + i0^+} + \frac{i}{u - M^2 + i0^+} \right) \\ &\stackrel{(*)}{=} \frac{3i\lambda^2}{M^2} \left[ 1 + \mathcal{O}\left(\frac{\{s, t, u\}}{M^2}\right) \right]. \end{aligned}$$

**Effective theory: Description in terms of contact interaction**



$$\mathcal{M}_{\text{eff}} = \frac{i\lambda^2 4!}{8M^2} = \frac{3i\lambda^2}{M^2}.$$

**Both calculations yield the same result!**

**EFT calculation simpler.**

- Weinberg's effective field theory program:<sup>2</sup>

... if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. ...

Explanation of terms:

- Symmetries: Poincaré invariance, discrete symmetries  $C$ ,  $P$ ,  $T$ , but also internal symmetries such as isospin symmetry, chiral symmetry including the possibility of a spontaneous symmetry breakdown.

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<sup>2</sup>S. Weinberg, *Physica A* 96, 327 (1979)

- Analyticity  $\leftrightarrow$  Causality.
- Unitarity: The sum over the probabilities of the final states must yield exactly 1:

$$\sum_f |\langle f|S|i\rangle|^2 = 1.$$

- Cluster decomposition:<sup>3</sup> Loosely speaking, distant experiments must yield uncorrelated results:

$$S_{\gamma+\delta\leftarrow\alpha+\beta} \rightarrow S_{\delta\leftarrow\beta}S_{\gamma\leftarrow\alpha}.$$

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<sup>3</sup>S. Weinberg, *The Quantum Theory Of Fields. Vol. 1: Foundations* (Cambridge University Press, Cambridge, 1995), chapter 4

- **Aim of these lectures:**

Most general description of the strong interactions at low energies:  $\pi\pi$ ,  $\pi N$ , etc.

- **Challenge:**

We need the

1. the most general Lagrangian;
2. a consistent power counting scheme to perform perturbative calculations.



# 1. QCD and chiral symmetry

## Some Remarks on SU(3)

### SU(3) in the context of the strong interactions

1. Gauge group of quantum chromodynamics (QCD)
2. Flavor SU(3)
3. Chiral symmetry for massless  $u$ ,  $d$ , and  $s$  quarks  
 $SU(3)_L \times SU(3)_R$

## Definition

$SU(3) \equiv$  set of all unitary, unimodular,  $3 \times 3$  matrices  $U$ :

$$U^\dagger U = 1, \det(U) = 1.$$

## Exponential representation

$$U(\Theta) = \exp \left( -i \sum_{a=1}^8 \Theta_a \frac{\lambda_a}{2} \right), \quad \Theta_a \text{ real numbers.}$$

## Gell-Mann matrices

$$\begin{aligned} \frac{\lambda_a}{2} &= i \frac{\partial U}{\partial \Theta_a} (0, \dots, 0), \\ \lambda_a &= \lambda_a^\dagger, \\ \text{Tr}(\lambda_a \lambda_b) &= 2\delta_{ab}, \\ \text{Tr}(\lambda_a) &= 0. \end{aligned}$$

## Explicit representation

$$\begin{aligned}\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.\end{aligned}$$

Set  $\{i\lambda_a\}$ : Basis of the Lie algebra  $\mathfrak{su}(3)$ .

Commutation relations

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = if_{abc}\frac{\lambda_c}{2}$$

Totally antisymmetric real structure constants

**Exercise:**  $f_{abc} = \frac{1}{4i}\text{Tr}([\lambda_a, \lambda_b]\lambda_c)$

$abc$	123	147	156	246	257	345	367	458	678
$f_{abc}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$

Anti-commutation relations

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c$$

Totally symmetric  $d_{abc}$

**Exercise:**  $d_{abc} = \frac{1}{4}\text{Tr}(\{\lambda_a, \lambda_b\}\lambda_c)$

## Introduce

$$\lambda_0 = \sqrt{2/3} \text{diag}(1, 1, 1)$$

Arbitrary  $3 \times 3$  matrix  $M$  can be written as

$$M = \sum_{a=0}^8 M_a \lambda_a$$

$M_a$  complex numbers

$$M_a = \frac{1}{2} \text{Tr}(\lambda_a M)$$

## The QCD Lagrangian

Quantum chromodynamics (QCD) is the gauge theory of the strong interactions with color SU(3) as the underlying gauge group

- Ingredients

The matter fields of QCD are the so-called quarks which are spin-1/2 fermions, with six different flavors in addition to their three possible colors

<b>flavor</b>	<b>u</b>	<b>d</b>	<b>s</b>
<b>charge [e]</b>	$2/3$	$-1/3$	$-1/3$
<b>mass [MeV]</b>	1.7 – 3.3	4.1 – 5.8	$101^{+29}_{-21}$
<b>flavor</b>	<b>c</b>	<b>b</b>	<b>t</b>
<b>charge [e]</b>	$2/3$	$-1/3$	$2/3$
<b>mass [GeV]</b>	$1.27^{+0.07}_{-0.09}$	$4.19^{+0.18}_{-0.06}$	$172.0 \pm 0.9 \pm 1.3$

## Quark field components

$$q_{\alpha,f,A}$$

$\alpha = 1, 2, 3, 4$ : Dirac spinor index

$f = 1, 2, 3, 4, 5, 6$ : flavor index (u,d,s,c,b,t)

$A = 1, 2, 3$ : color index (red,green,blue)

QCD Lagrangian (apply the gauge principle with respect to the group SU(3))

Long version

$$\mathcal{L}_{\text{QCD}} = \sum_{f,f'=1}^6 \sum_{A,A'=1}^3 \sum_{\alpha,\alpha'=1}^4 \bar{q}_{\alpha,f,A} \left[ (\gamma_{\alpha\alpha'}^{\mu} i \partial_{\mu} - m_f \delta_{\alpha\alpha'}) \delta_{AA'} - g_3 \sum_{a=1}^8 \mathcal{A}_{\mu,a} \frac{\lambda_{a,AA'}}{2} \gamma_{\alpha\alpha'}^{\mu} \right] \delta_{ff'} q_{\alpha',f',A'} - \sum_{a=1}^8 \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}_a^{\mu\nu}$$

from gauge principle

## Short version

$$\mathcal{L}_{\text{QCD}} = \sum_{f=\substack{u,d,s, \\ c,b,t}} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}_a^{\mu\nu}.$$

## Extremely short version

$$\mathcal{L}_{\text{QCD}} = \bar{q} (i\not{D} - \mathcal{M}) q - \frac{1}{2} \text{Tr}_c (\mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu})$$

## Gauge transformation of the quark fields

$$q_f = \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix} \mapsto q'_f = \exp \left[ -i \sum_{a=1}^8 \Theta_a(x) \frac{\lambda_a}{2} \right] q_f = U[\Theta(x)] q_f$$

## Transformation behavior of the gauge fields

$$\mathcal{A}_\mu \equiv \mathcal{A}_{\mu,a} \frac{\lambda_a}{2} \mapsto U \mathcal{A}_\mu U^\dagger + \frac{i}{g_3} \partial_\mu U U^\dagger$$



Covariant derivative of the quark fields

$$D_\mu q_f = (\partial_\mu + ig_3 \mathcal{A}_\mu) q_f \quad \xrightarrow{\text{Exercise:}} \quad (D_\mu q_f)' = D'_\mu q'_f = U D_\mu q_f$$

transforms as quark fields

Field strengths transform as

$$\mathcal{G}_{\mu\nu} \equiv \mathcal{G}_{\mu\nu,a} \frac{\lambda_a}{2} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + ig_3 [\mathcal{A}_\mu, \mathcal{A}_\nu] \quad \xrightarrow{\text{Exercise:}} \quad U \mathcal{G}_{\mu\nu} U^\dagger$$

Equivalent (Gell-Mann matrices!)

$$\mathcal{G}_{\mu\nu,a} = \partial_\mu \mathcal{A}_{\nu,a} - \partial_\nu \mathcal{A}_{\mu,a} - g_3 f_{abc} \mathcal{A}_{\mu,b} \mathcal{A}_{\nu,c}$$

**Exercise:**  $\mathcal{L}_{\text{QCD}}$  invariant under local SU(3)

Gauge invariance also allows for

$$\begin{aligned}\mathcal{L}_\theta &= \frac{g_3^2 \bar{\theta}}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} \sum_{a=1}^8 \mathcal{G}_{\mu\nu}^a \mathcal{G}_{\rho\sigma}^a \\ &= \frac{g_3^2 \bar{\theta}}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr}_c(\mathcal{G}_{\mu\nu} \mathcal{G}_{\rho\sigma})\end{aligned}$$

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ even permutation of } \{0, 1, 2, 3\} \\ -1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ odd permutation of } \{0, 1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

So-called  $\theta$  term implies explicit  $P$  and  $CP$  violation in the strong interactions

$\rightsquigarrow$ , e.g., electric dipole moment of the neutron

empirical information: very small

## Accidental, Global Symmetries of $\mathcal{L}_{\text{QCD}}$

The pion is special!

quark content	mesons
$u\bar{d}$	$\pi^+, \rho^+$
$(u\bar{u} - d\bar{d})/\sqrt{2}$	$\pi^0, \rho^0$
$d\bar{u}$	$\pi^-, \rho^-$

$$M_{\pi^+} = 140 \text{ MeV} \ll M_{\rho} = 776 \text{ MeV},$$
$$M_{\pi} \ll m_p = 938 \text{ MeV}.$$

$$M_{\pi^+} < M_{K^+} = 494 \text{ MeV},$$
$$M_{\pi^+} \ll M_{D^+} = 1869 \text{ MeV}.$$

$$\begin{pmatrix} m_u = 0.0023 \text{ GeV} \\ m_d = 0.0048 \text{ GeV} \\ m_s = 0.95 \text{ GeV} \end{pmatrix} \ll \Lambda_\chi \approx 1 \text{ GeV} \leq \begin{pmatrix} m_c = 1.275 \text{ GeV} \\ m_b = 4.18 \text{ GeV} \\ m_t = 173.5 \text{ GeV} \end{pmatrix}$$

## Motivation

$$m_p \gg 2m_u + m_d$$

Consider the light-flavor quarks in the so-called chiral limit  $m_u, m_d, m_s \rightarrow 0$  as starting point in the discussion of low-energy QCD:

$$\mathcal{L}_{\text{QCD}}^0 = \sum_{l=u,d,s} \bar{q}_l i \not{D} q_l - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}_a^{\mu\nu}$$

What are the global symmetries of  $\mathcal{L}_{\text{QCD}}^0$ ?

## Chirality matrix

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger, \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5^2 = 1$$

## Projection operators

$$P_L = \frac{1}{2}(1 - \gamma_5) = P_L^\dagger, \quad P_R = \frac{1}{2}(1 + \gamma_5) = P_R^\dagger$$

## Exercise: Properties

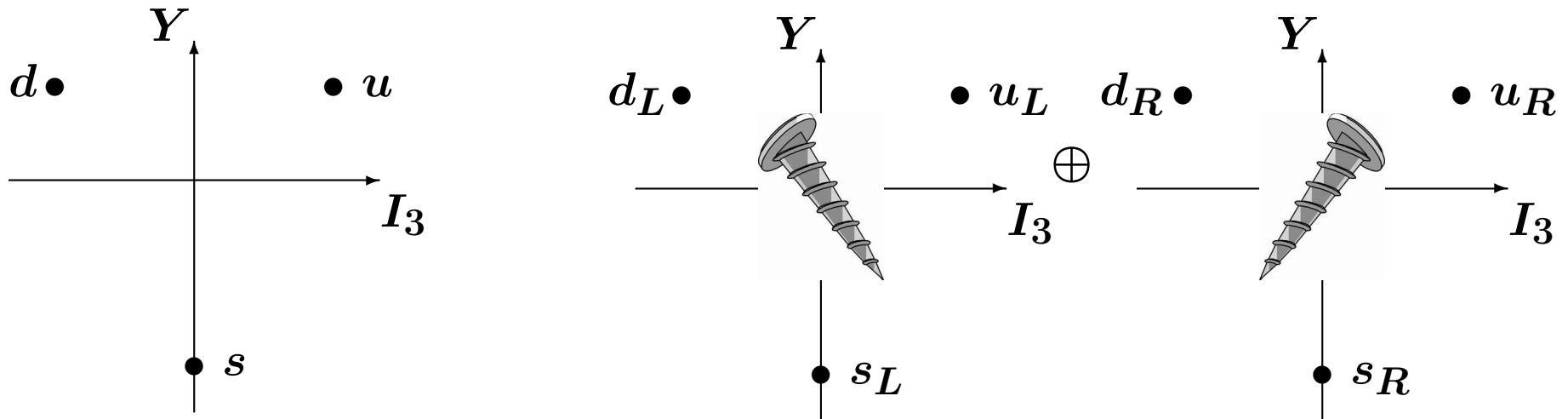
$$P_L + P_R = 1$$

$$P_L^2 = P_L, \quad P_R^2 = P_R$$

$$P_L P_R = P_R P_L = 0$$

## Left- and right-handed quark fields $q_L$ and $q_R$

$$q_L = P_L q, \quad q_R = P_R q$$



**Exercise:**

$$\bar{q}\Gamma_i q = \begin{cases} \bar{q}_L \Gamma_1 q_L + \bar{q}_R \Gamma_1 q_R & \text{for } \Gamma_1 \in \{\gamma^\mu, \gamma^\mu \gamma_5\} \\ \bar{q}_R \Gamma_2 q_L + \bar{q}_L \Gamma_2 q_R & \text{for } \Gamma_2 \in \{1, \gamma_5, \sigma^{\mu\nu}\} \end{cases}$$

$$(\bar{q}_L = \bar{q} P_R \text{ and } \bar{q}_R = \bar{q} P_L)$$

## QCD Lagrangian in the chiral limit

$$\mathcal{L}_{\text{QCD}}^0 = \sum_{l=u,d,s} (\bar{q}_{L,l} i \not{D} q_{L,l} + \bar{q}_{R,l} i \not{D} q_{R,l}) - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}_a^{\mu\nu}$$

invariant under (covariant derivative flavor independent!)

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp \left( -i \sum_{a=1}^8 \Theta_a^L \frac{\lambda_a}{2} \right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$
$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp \left( -i \sum_{a=1}^8 \Theta_a^R \frac{\lambda_a}{2} \right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

$\mathcal{L}_{\text{QCD}}^0$  has a classical global  $\mathbf{U}(3)_L \times \mathbf{U}(3)_R$  symmetry

Applying Noether's theorem from such an invariance one would expect a total of  $2 \times (8 + 1) = 18$  conserved currents

## Noether's Theorem

Continuous symmetries  $\leftrightarrow$  conserved quantities

Lagrangian  $\mathcal{L}$  depending on  $n$  independent fields  $\Phi_i$

$$\mathcal{L} = \mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$$

$n$  equations of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \dots, n$$

Consider transformations which depend on  $r$  real local parameters  $\epsilon_a(x)$  (method of Gell-Mann and Lévy)

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) + \delta \Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]$$



## Variation of the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}(\Phi'_i, \partial_\mu\Phi'_i) - \mathcal{L}(\Phi_i, \partial_\mu\Phi_i) = \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu\delta\Phi_i$$

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$$\partial_\mu\delta\Phi_i = -i[\partial_\mu\epsilon_a(x)]F_i^a - i\epsilon_a(x)\partial_\mu F_i^a$$

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$$\begin{aligned}\dots &= \epsilon_a(x) \left( -i\frac{\partial\mathcal{L}}{\partial\Phi_i}F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a \right) + \partial_\mu\epsilon_a(x) \left( -i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}F_i^a \right) \\ &\equiv \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu\epsilon_a(x)J^{\mu,a}\end{aligned}$$

## Consistency for solutions to the EOM

$$\partial_\mu J^{\mu,a} = -i \left( \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \right) F_i^a - i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \partial_\mu F_i^a = -i \frac{\partial\mathcal{L}}{\partial\Phi_i} F_i^a - i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \partial_\mu F_i^a$$

## Currents and divergences of currents from variation

$$J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \partial_\mu \epsilon_a}$$
$$\partial_\mu J^{\mu,a} = \frac{\partial \delta \mathcal{L}}{\partial \epsilon_a}$$

Assume Lagrangian to be invariant under a **global** transformation:

$$\delta \mathcal{L} = 0 \quad \wedge \quad \partial_\mu \epsilon_a(x) J^{\mu,a} = 0$$

⇒ Current  $J^{\mu,a}$  is conserved

$$\partial_\mu J^{\mu,a} = 0$$

Charge

$$Q^a(t) = \int d^3x J_0^a(t, \vec{x})$$

is time independent, i.e., a constant of the motion

## Transition to a Quantum (Field) Theory

**Analogy:** Point mass  $m$  in a central potential  $V(\vec{r}) = V(r)$ . Lagrange and Hamilton functions are rotationally invariant.

**Consequence:** Angular momentum  $\vec{l} = \vec{r} \times \vec{p}$  is a constant of the motion.

Transition to quantum mechanics.  $\Rightarrow$  Operators

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0.$$

Components of the angular momentum operator

$$\hat{l}_i = \epsilon_{ijk} \hat{x}_j \hat{p}_k = -i\hat{p}_j (-i\epsilon_{ijk}) \hat{x}_k$$

satisfy the commutation relations

$$[\hat{l}_i, \hat{l}_j] = i\epsilon_{ijk} \hat{l}_k$$

i.e., they cannot simultaneously be diagonalized.

**Angular momentum operators are generators of rotations:**

$$|\Psi'\rangle = \exp(-i\alpha_i \hat{l}_i) |\Psi\rangle.$$

**Rotational invariance of the quantum system**

$$[\hat{H}, \hat{l}_i] = 0$$

**i.e.,  $\hat{l}_i$  are still constants of the motion.**

**Simultaneously diagonalize  $\hat{H}$ ,  $\vec{\hat{l}}$ , and  $\hat{l}_3$ .**

**Multiplets with eigenvalues  $l(l+1)$  and  $m = -l, \dots, l$   
( $l = 0, 1, 2, \dots$ ).**

**Energy eigenvalues depend on  $V$  (dynamics).**

**Classify operators according to transformation behavior.**

**Example: Components  $\hat{A}_i$  of a vector operator**

$$[\hat{l}_i, \hat{A}_j] = i\epsilon_{ijk} \hat{A}_k.$$

**Use Wigner-Eckart theorem to calculate matrix elements.**

## Analogous case in quantum field theory

Canonical quantization: Fields  $\Phi_i$  and their conjugate momenta  $\Pi_i = \partial\mathcal{L}/\partial(\partial_0\Phi_i) \Rightarrow$  operators.

$$\begin{aligned} [\Phi_i(t, \vec{x}), \Pi_j(t, \vec{y})] &= i\delta^3(\vec{x} - \vec{y})\delta_{ij} \leftrightarrow [\hat{x}_i, \hat{p}_j] = i\delta_{ij} \\ [\Phi_i(t, \vec{x}), \Phi_j(t, \vec{y})] &= 0 \leftrightarrow [\hat{x}_i, \hat{x}_j] = 0 \\ [\Pi_i(t, \vec{x}), \Pi_j(t, \vec{y})] &= 0 \leftrightarrow [\hat{p}_i, \hat{p}_j] = 0 \end{aligned}$$

Consider infinitesimal transformations which are linear in the fields,

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) - i\epsilon_a(x)t_{ij}^a\Phi_j(x) \quad \leftrightarrow \quad \hat{x}_i \mapsto \hat{x}_i - i\epsilon_k(-i\epsilon_{kij})\hat{x}_j$$

$t_{ij}^a$  are constants generating a mixing of the fields

$$J^{\mu,a}(x) = -it_{ij}^a \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \Phi_j$$

$$Q^a(t) = -i \int d^3x \Pi_i(x) t_{ij}^a \Phi_j(x) \quad \leftrightarrow \quad \hat{l}_k = -i\hat{p}_i(-i\epsilon_{kij})\hat{x}_j = \epsilon_{kij}\hat{x}_i\hat{p}_j$$

where  $J^{\mu,a}(x)$  and  $Q^a(t)$  are now operators

Transformation behavior of field operators

$$\begin{aligned} [Q^a(t), \Phi_k(t, \vec{y})] &= -it_{ij}^a \int d^3x [\Pi_i(t, \vec{x}) \Phi_j(t, \vec{x}), \Phi_k(t, \vec{y})] \\ &= -t_{kj}^a \Phi_j(t, \vec{y}) \\ \leftrightarrow [\hat{l}_k, \hat{x}_i] &= i\epsilon_{kij} \hat{x}_j \end{aligned}$$

$Q^a$  are generators of the transformations acting on the states of Hilbert space

## Global Symmetry Currents of the Light Quark Sector

Consider infinitesimal, local transformations

$$\begin{aligned} q_L &\mapsto \left( 1 - i \sum_{a=1}^8 \epsilon_a^L \frac{\lambda_a}{2} - i\epsilon^L \right) q_L \\ q_R &\mapsto \dots \end{aligned}$$

Variation

$$\delta \mathcal{L}_{\text{QCD}}^0 = \bar{q}_L \left( \sum_{a=1}^8 \partial_\mu \epsilon_a^L \frac{\lambda_a}{2} + \partial_\mu \epsilon^L \right) \gamma^\mu q_L + (L \rightarrow R)$$

Currents

$$\begin{aligned} L^{\mu,a} &= \frac{\partial \delta \mathcal{L}_{\text{QCD}}^0}{\partial \partial_\mu \epsilon_a^L} = \bar{q}_L \gamma^\mu \frac{\lambda_a}{2} q_L, & \partial_\mu L^{\mu,a} &= \frac{\partial \delta \mathcal{L}_{\text{QCD}}^0}{\partial \epsilon_a^L} = 0 \\ L^\mu &= \frac{\partial \delta \mathcal{L}_{\text{QCD}}^0}{\partial \partial_\mu \epsilon^L} = \bar{q}_L \gamma^\mu q_L, & \partial_\mu L^\mu &= \frac{\partial \delta \mathcal{L}_{\text{QCD}}^0}{\partial \epsilon^L} = 0 \end{aligned}$$

+ analogous expressions for  $R^{\mu,a}$  and  $R^\mu$

## Linear combinations

$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q$$

$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q$$

Vector and axial-vector current densities, respectively,

$$P : V^{\mu,a}(t, \vec{x}) \mapsto V_\mu^a(t, -\vec{x})$$

$$P : A^{\mu,a}(t, \vec{x}) \mapsto -A_\mu^a(t, -\vec{x})$$

Conserved singlet vector current

$$V^\mu = R^\mu + L^\mu = \bar{q}\gamma^\mu q, \quad \partial_\mu V^\mu = 0$$

Singlet axial-vector current

$$A^\mu = R^\mu - L^\mu = \bar{q}\gamma^\mu \gamma_5 q$$

This symmetry is not preserved by quantization and there will be extra terms, referred to as anomalies, resulting in

$$\partial_\mu A^\mu = \frac{3g_3^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\rho\sigma}, \quad \epsilon_{0123} = 1$$



## The Chiral Algebra

Define “charge operators” as the space integrals of the charge densities

$$Q_L^a(t) = \int d^3x q_L^\dagger(t, \vec{x}) \frac{\lambda^a}{2} q_L(t, \vec{x}), \quad a = 1, \dots, 8$$

$$Q_R^a(t) = \int d^3x q_R^\dagger(t, \vec{x}) \frac{\lambda^a}{2} q_R(t, \vec{x}), \quad a = 1, \dots, 8$$

$$Q_V(t) = \int d^3x \left[ q_L^\dagger(t, \vec{x}) q_L(t, \vec{x}) + q_R^\dagger(t, \vec{x}) q_R(t, \vec{x}) \right]$$

$H_{\text{QCD}}^0$  exhibits a global  $\text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V$  symmetry

$$[Q_L^a, H_{\text{QCD}}^0] = [Q_R^a, H_{\text{QCD}}^0] = [Q_V, H_{\text{QCD}}^0] = 0$$

Lie algebra of  $\text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V$

$$[Q_L^a, Q_L^b] = if_{abc} Q_L^c$$

$$[Q_R^a, Q_R^b] = if_{abc} Q_R^c$$

$$[Q_L^a, Q_R^b] = 0$$

$$[Q_L^a, Q_V] = [Q_R^a, Q_V] = 0$$

How does one verify the commutation relations of the charge operators?

### 1. Anti-commutation relations of Fermi fields

$$\begin{aligned} \{q_{f,A,\alpha}(t, \vec{x}), q_{f',A',\alpha'}^\dagger(t, \vec{y})\} &= \delta^3(\vec{x} - \vec{y})\delta_{ff'}\delta_{AA'}\delta_{\alpha\alpha'} \\ \{q_{f,A,\alpha}(t, \vec{x}), q_{f',A',\alpha'}(t, \vec{y})\} &= 0 \\ \{q_{f,A,\alpha}^\dagger(t, \vec{x}), q_{f',A',\alpha'}^\dagger(t, \vec{y})\} &= 0 \end{aligned}$$

**2. Exercise:**  $[ab, cd] = a\{b, c\}d - ac\{b, d\} + \{a, c\}db - c\{a, d\}b$

**3. Let  $F_i$ ,  $C_i$ , and  $\Gamma_i$  be  $3 \times 3$  flavor matrices,  $3 \times 3$  color matrices,  $4 \times 4$  Dirac matrices, respectively**

$$\begin{aligned} [q^\dagger(t, \vec{x})F_1C_1\Gamma_1q(t, \vec{x}), q^\dagger(t, \vec{y})F_2C_2\Gamma_2q(t, \vec{y})] &= \\ \delta^3(\vec{x} - \vec{y})q^\dagger(t, \vec{x})F_1F_2C_1C_2\Gamma_1\Gamma_2q(t, \vec{y}) & \\ -\delta^3(\vec{x} - \vec{y})q^\dagger(t, \vec{y})F_2F_1C_2C_1\Gamma_2\Gamma_1q(t, \vec{x}) & \end{aligned}$$

4. Insert appropriate projection operators

5. Integrate with respect to  $\vec{x}$  and  $\vec{y}$

Example (recall  $P_L^\dagger = P_L$  and  $P_L^2 = P_L$ )

$$\begin{aligned} [Q_L^a, Q_L^b] &= \int d^3x d^3y [q^\dagger(t, \vec{x}) P_L^\dagger \frac{\lambda_a}{2} P_L q(t, \vec{x}), q^\dagger(t, \vec{y}) P_L^\dagger \frac{\lambda_b}{2} P_L q(t, \vec{y})] \\ &= \int d^3x d^3y \delta^3(\vec{x} - \vec{y}) q^\dagger(t, \vec{x}) \underbrace{P_L^\dagger P_L P_L^\dagger P_L}_{P_L} \frac{\lambda_a}{2} \frac{\lambda_b}{2} q(t, \vec{y}) \\ &\quad - \int d^3x d^3y \delta^3(\vec{x} - \vec{y}) q^\dagger(t, \vec{y}) P_L \frac{\lambda_b}{2} \frac{\lambda_a}{2} q(t, \vec{x}) \\ &= i f_{abc} \int d^3x q^\dagger(t, \vec{x}) \frac{\lambda_c}{2} P_L q(t, \vec{x}) = i f_{abc} Q_L^c \end{aligned}$$

## Chiral Symmetry Breaking by the Quark Masses

$$\mathcal{M} = \text{diag}(m_u, m_d, m_s)$$

Quark-mass term mixes left- and right-handed fields

$$\mathcal{L}_{\mathcal{M}} = -\bar{q}\mathcal{M}q = -(\bar{q}_R\mathcal{M}q_L + \bar{q}_L\mathcal{M}q_R)$$

Transformation of left-handed fields

$$q_L \mapsto \left( 1 - i \sum_{a=1}^8 \epsilon_a^L \frac{\lambda_a}{2} - i\epsilon^L \right) q_L$$

Variation  $\delta\mathcal{L}_{\mathcal{M}}$

$$\begin{aligned} \delta\mathcal{L}_{\mathcal{M}} &= - \left[ -i\bar{q}_R\mathcal{M} \left( \sum_{a=1}^8 \epsilon_a^L \frac{\lambda_a}{2} + \epsilon^L \right) q_L + i\bar{q}_L \left( \sum_{a=1}^8 \epsilon_a^L \frac{\lambda_a}{2} + \epsilon^L \right) \mathcal{M}q_R \right] \\ &= -i \left[ \sum_{a=1}^8 \epsilon_a^L \left( \bar{q}_L \frac{\lambda_a}{2} \mathcal{M}q_R - \bar{q}_R \mathcal{M} \frac{\lambda_a}{2} q_L \right) + \epsilon^L (\bar{q}_L \mathcal{M}q_R - \bar{q}_R \mathcal{M}q_L) \right] \end{aligned}$$

## Divergences

$$\partial_\mu L^{\mu,a} = \frac{\partial \delta \mathcal{L}_{\mathcal{M}}}{\partial \epsilon_a^L} = -i \left( \bar{q}_L \frac{\lambda_a}{2} \mathcal{M} q_R - \bar{q}_R \mathcal{M} \frac{\lambda_a}{2} q_L \right)$$

$$\partial_\mu L^\mu = \frac{\partial \delta \mathcal{L}_{\mathcal{M}}}{\partial \epsilon^L} = -i (\bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L)$$

+ analogous expressions for  $\partial_\mu R^{\mu,a}$  and  $\partial_\mu R^\mu$  ( $R \leftrightarrow L$ )

## More common (linear combinations)

$$\partial_\mu V^{\mu,a} = i\bar{q} \left[ \mathcal{M}, \frac{\lambda_a}{2} \right] q$$

$$\partial_\mu A^{\mu,a} = i\bar{q} \left\{ \frac{\lambda_a}{2}, \mathcal{M} \right\} \gamma_5 q$$

$$\partial_\mu V^\mu = 0$$

$$\partial_\mu A^\mu = 2i\bar{q} \mathcal{M} \gamma_5 q + \frac{3g_3^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathcal{G}_a^{\mu\nu} \mathcal{G}_a^{\rho\sigma}, \quad \epsilon_{0123} = 1$$

## Summary

- Massless quarks: **16** conserved currents  $L^{\mu,a}$  and  $R^{\mu,a}$  ( $V^{\mu,a}$  and  $A^{\mu,a}$ ) + **1** conserved singlet vector current  $V^\mu$ . Singlet axial-vector current  $A^\mu$  has an **anomaly**.
- For any value of quark masses: flavor currents  $\bar{u}\gamma^\mu u$ ,  $\bar{d}\gamma^\mu d$ , and  $\bar{s}\gamma^\mu s$  are always conserved.
- Equal quark masses  $m_u = m_d = m_s$ :
  - **8** conserved vector currents  $V^{\mu,a}$  ( $[\lambda_a, 1] = 0$ ).  
SU(3) flavor symmetry.
  - 8 axial-vector currents  $A^{\mu,a}$  are not conserved.  
Microscopic origin of the PCAC relation (partially conserved axial-vector current).
- $m_u = m_d$ : isospin symmetry.

## Green Functions and Chiral Ward Identities

### Symmetry currents

$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q$$

$$V^\mu = \bar{q}\gamma^\mu q$$

$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q$$

+ scalar and pseudoscalar densities (see divergences of currents)

$$S_a(x) = \bar{q}(x)\lambda_a q(x)$$

$$P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x)$$

Green functions: Matrix elements of time-ordered products

Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism: Relation to scattering amplitudes

## Examples

### “Vacuum” sector

$\langle 0 T[A_a^\mu(x)P_b(y)] 0\rangle$	pion decay
$\langle 0 T[P_a(x)J^\mu(y)P_c(z)] 0\rangle$	pion electromagnetic form factor
$\langle 0 T[P_a(w)P_b(x)P_c(y)P_d(z)] 0\rangle$	pion-pion scattering

### One-nucleon sector

$\langle N J^\mu(x) N\rangle$	nucleon electromagnetic form factors
$\langle N A_a^\mu(x) N\rangle$	axial form factor + ...
$\langle N T[J^\mu(x)J^\nu(y)] N\rangle$	Compton scattering
$\langle N T[J^\mu(x)P_a(y)] N\rangle$	pion electroproduction

**A chiral Ward identity** relates the divergence of a Green function containing at least one factor of  $V^{\mu,a}$  or  $A^{\mu,a}$  to some linear combination of other Green functions.

**Q:** Why chiral?

**A:**  $V^{\mu,a}$  and  $A^{\mu,a}$  contain  $L^{\mu,a}$  and  $R^{\mu,a}$



## Simple example

$$\begin{aligned} G_{AP}^{\mu,ab}(x,y) &= \langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle \\ &= \Theta(x_0 - y_0)\langle 0|A_a^\mu(x)P_b(y)|0\rangle \\ &\quad + \Theta(y_0 - x_0)\langle 0|P_b(y)A_a^\mu(x)|0\rangle \end{aligned}$$

## Divergence

$$\begin{aligned} \partial_\mu^x G_{AP}^{\mu,ab}(x,y) &= \partial_\mu^x [\Theta(x_0 - y_0)\langle 0|A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)A_a^\mu(x)|0\rangle] \\ &= \delta(x_0 - y_0)\langle 0|A_a^0(x)P_b(y)|0\rangle - \delta(x_0 - y_0)\langle 0|P_b(y)A_a^0(x)|0\rangle \\ &\quad + \Theta(x_0 - y_0)\langle 0|\partial_\mu^x A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)\partial_\mu^x A_a^\mu(x)|0\rangle \\ &= \delta(x_0 - y_0)\langle 0|[A_a^0(x), P_b(y)]|0\rangle + \langle 0|T[\partial_\mu^x A_a^\mu(x)P_b(y)]|0\rangle \end{aligned}$$

We made use of

$$\partial_\mu^x \Theta(x_0 - y_0) = \delta(x_0 - y_0)g_{0\mu} = -\partial_\mu^x \Theta(y_0 - x_0)$$

## Main features of (chiral) Ward identities:

1. Differentiation of the theta functions  $\Rightarrow$  Equal-time commutators between a charge density and the remaining quadratic forms  $\Rightarrow$  Reflection of underlying symmetry
2. Divergence of the current operator in question.
  - **Perfect symmetry**  $\Rightarrow$  such terms vanish  
Example: Electromagnetic case with its U(1) symmetry.
  - **Approximate symmetry**  $\Rightarrow$  additional term involving the symmetry breaking appears  
For a soft breaking such a divergence can be treated as a perturbation.

## QCD in the Presence of External Fields and the Generating Functional

So far: Explicitly work out the chiral Ward identity you are interested in.

Q: Is it possible to somehow obtain **all** chiral Ward identities from a single expression?

A: Yes (without proof)

- Introduce into the Lagrangian of QCD the couplings of the
  1. nine vector currents
  2. eight axial-vector currents
  3. nine scalar quark densities

#### 4. nine pseudoscalar quark densities

to external c-number fields  $v^\mu(x)$ ,  $v_{(s)}^\mu$ ,  $a^\mu(x)$ ,  $s(x)$ , and  $p(x)$ :

$$\mathcal{L} = \mathcal{L}_{\text{QCD}}^0 + \mathcal{L}_{\text{ext}}$$

$$\mathcal{L}_{\text{ext}} = \bar{q}\gamma_\mu(v^\mu + \frac{1}{3}v_{(s)}^\mu + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q.$$

#### Parameterization

$$v^\mu = \sum_{a=1}^8 \frac{\lambda_a}{2} v_a^\mu, \quad a^\mu = \sum_{a=1}^8 \frac{\lambda_a}{2} a_a^\mu, \quad s = \sum_{a=0}^8 \lambda_a s_a, \quad p = \sum_{a=0}^8 \lambda_a p_a.$$

- Combine all Green functions in a generating functional

$$\exp(iZ[v, a, s, p]) = \langle 0 | T \exp \left[ i \int d^4x \mathcal{L}_{\text{ext}}(x) \right] | 0 \rangle$$

- Obtain Green function through a functional derivative with respect to the external fields

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## Pedagogical illustration

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0(\phi) + \mathcal{L}_{\text{ext}}, \\ \mathcal{L}_{\text{ext}} &= j(x)\phi(x)\end{aligned}$$

Generating functional for Green functions of the type

$$G(x_1, \dots, x_n) = \langle 0 | T[\phi(x_1) \cdots \phi(x_n)] | 0 \rangle$$

$$\begin{aligned}\exp(iZ[j]) &= \langle 0 | T \exp \left[ i \int d^4x \mathcal{L}_{\text{ext}}(x) \right] | 0 \rangle \\ &= 1 + i \int d^4x j(x) \langle 0 | \phi(x) | 0 \rangle \\ &\quad + \sum_{k=2} \frac{i^k}{k!} \int d^4x_1 \cdots d^4x_k j(x_1) \cdots j(x_k) \langle 0 | T[\phi(x_1) \cdots \phi(x_k)] | 0 \rangle \\ &= \cdots + \frac{i^2}{2} \int d^4x_1 d^4x_2 j(x_1) j(x_2) \langle 0 | T[\phi(x_1) \phi(x_2)] | 0 \rangle + \cdots\end{aligned}$$

E.g.

$$\begin{aligned} G(x_1, x_2) &= \langle 0 | T[\phi(x_1)\phi(x_2)] | 0 \rangle \\ &= (-i)^2 \frac{\delta^2 Z[j]}{\delta j(x_1)\delta j(x_2)} \Big|_{j=0} \end{aligned}$$

**Powers and sort of functional derivatives must match:**

$$1, \quad i \int d^4x j(x) \langle 0 | \phi(x) | 0 \rangle : \quad \text{too few terms}$$

$$\frac{i^k}{k!} \int d^4x_1 \cdots d^4x_k j(x_1) \cdots j(x_k) \langle 0 | \phi(x_1) \cdots \phi(x_k) | 0 \rangle, k \geq 3 :$$

**too many terms, because  $j$  is set equal to 0 at the end**

**Exercise:** Make use of

$$\frac{\delta j(x)}{\delta j(y)} = \delta^4(x - y)$$

$$\frac{\delta^2}{\delta j(x_1)\delta j(x_2)} \frac{1}{2} \int d^4x d^4y j(x)j(y) \langle 0|T[\phi(x)\phi(y)]|0\rangle = \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle$$


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- **Examples**

- **Scalar quark condensate in the chiral limit,  $\langle 0|\bar{u}u|0\rangle_0$ ,**

$$\langle 0|\bar{u}(x)u(x)|0\rangle_0 =$$

$$\frac{i}{2} \left[ \sqrt{\frac{2}{3}} \frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \frac{1}{\sqrt{3}} \frac{\delta}{\delta s_8(x)} \right] \exp(iZ[v, a, s, p]) \Big|_{v=a=s=p=0}$$

**Subscript 0: Chiral limit**

- **Two-point function of two axial-vector currents of the “real world,” i.e., for  $s = \text{diag}(m_u, m_d, m_s)$ , and the “true vacuum”**

$|0\rangle,$

$$\langle 0|T[A_\mu^a(x)A_\nu^b(0)]|0\rangle =$$

$$(-i)^2 \frac{\delta^2}{\delta a_a^\mu(x) \delta a_b^\nu(0)} \exp(iZ[v, a, s, p]) \Big|_{v=a=p=0, s=\text{diag}(m_u, m_d, m_s)} .$$

- **Q: But where is QCD?**

**A: In  $|0\rangle$  and  $q$  (solutions to EOM)**

**(The actual value of the generating functional for a given configuration of external fields  $v$ ,  $a$ ,  $s$ , and  $p$  reflects the dynamics generated by the QCD Lagrangian.)**

- **Q: But where is the (infinite) set of all chiral Ward identities?**

**A: Ward identities obeyed by the Green functions are equivalent to an invariance of the generating functional under a **local** transformation of the external fields**



- The use of local transformations allows one to also consider divergences of Green functions.
- Q: What do we require of the external fields?  
A: We want  $\mathcal{L}$  to be a Hermitian Lorentz scalar, to be even under  $P$ ,  $C$ , and  $T$ , and to be invariant under **local** chiral transformations.

What does that imply for the external fields?

- Parity

Transformation behavior of quark fields

$$q_f(t, \vec{x}) \xrightarrow{P} \gamma^0 q_f(t, -\vec{x})$$

Properties of the Dirac matrices  $\Gamma$

$\Gamma$	1	$\gamma^\mu$	$\sigma^{\mu\nu}$	$\gamma_5$	$\gamma^\mu \gamma_5$
$\gamma_0 \Gamma \gamma_0$	1	$\gamma_\mu$	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu \gamma_5$

Requirement of parity conservation

$$\mathcal{L}(t, \vec{x}) \xrightarrow{P} \mathcal{L}(t, -\vec{x})$$

$\Rightarrow$

$$v^\mu \xrightarrow{P} v_\mu, \quad v_{(s)}^\mu \xrightarrow{P} v_\mu^{(s)}, \quad a^\mu \xrightarrow{P} -a_\mu, \quad s \xrightarrow{P} s, \quad p \xrightarrow{P} -p.$$

(Change of arguments from  $(t, \vec{x})$  to  $(t, -\vec{x})$  implied.)

**Example:**

$$\bar{q}(t, \vec{x}) \gamma^\mu v_\mu(t, \vec{x}) q(t, \vec{x}) \xrightarrow{P} \bar{q}(t, -\vec{x}) \gamma^0 \gamma^\mu \tilde{v}_\mu(t, -\vec{x}) \gamma^0 q(t, -\vec{x})$$

**Tilde denotes the transformed external field.**

**Make use of table, i.e.,  $\gamma^0 \gamma^\mu \gamma^0 = \gamma_\mu$ ,**

$$\dots = \bar{q}(t, -\vec{x}) \gamma_\mu \tilde{v}_\mu(t, -\vec{x}) q(t, -\vec{x}) \stackrel{!}{=} \bar{q}(t, -\vec{x}) \gamma_\mu v^\mu(t, -\vec{x}) q(t, -\vec{x}).$$

**We thus obtain**

$$v_\mu(t, \vec{x}) \xrightarrow{P} v^\mu(t, -\vec{x}).$$

- Charge conjugation

Transformation behavior of the quark fields

$$q_{\alpha,f} \xrightarrow{C} C_{\alpha\beta} \bar{q}_{\beta,f}, \quad \bar{q}_{\alpha,f} \xrightarrow{C} -q_{\beta,f} C_{\beta\alpha}^{-1},$$

$\alpha$  and  $\beta$ : Dirac spinor indices,

$f$ : flavor index

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -C^{-1} = -C^\dagger = -C^T$$

usual charge conjugation matrix

Properties of the Dirac matrices  $\Gamma$

$\Gamma$	1	$\gamma^\mu$	$\sigma^{\mu\nu}$	$\gamma_5$	$\gamma^\mu\gamma_5$
$-C\Gamma^T C$	1	$-\gamma^\mu$	$-\sigma^{\mu\nu}$	$\gamma_5$	$\gamma^\mu\gamma_5$

Using

$$\begin{aligned}
 \bar{q}\Gamma F q &= \bar{q}_{\alpha,f}\Gamma_{\alpha\beta}F_{ff'}q_{\beta,f'} \\
 &\stackrel{C}{\mapsto} -q_{\gamma,f}C_{\gamma\alpha}^{-1}\Gamma_{\alpha\beta}F_{ff'}C_{\beta\delta}\bar{q}_{\delta,f'} \\
 \text{Fermi statistics} &= \bar{q}_{\delta,f'}\underbrace{F_{ff'}}_{F_{f'f}^T}\underbrace{C_{\gamma\alpha}^{-1}\Gamma_{\alpha\beta}C_{\beta\delta}}_{(C^{-1}\Gamma C)^T}q_{\gamma,f} \\
 &= \bar{q}F^T \underbrace{(C^{-1}\Gamma C)^T}_{C^T\Gamma^T C^{-1T}} q \\
 &= -\bar{q}C\Gamma^T C F^T q
 \end{aligned}$$

Invariance of  $\mathcal{L}_{\text{ext}}$  under charge conjugation requires the transformation properties

$$v_{\mu} \xrightarrow{C} -v_{\mu}^T, \quad v_{\mu}^{(s)} \xrightarrow{C} -v_{\mu}^{(s)T}, \quad a_{\mu} \xrightarrow{C} a_{\mu}^T, \quad s, p \xrightarrow{C} s^T, p^T,$$

transposition refers to the flavor space.

- Time reversal: Nothing new

- Local chiral  $SU(3)_L \times SU(3)_R \times U(1)_V$  transformations

First step: Rewrite in terms of the left- and right-handed quark fields.

### Exercise

We first define

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu.$$

1. Make use of the projection operators  $P_L$  and  $P_R$  and verify

$$\begin{aligned} \bar{q}\gamma^\mu(v_\mu + \frac{1}{3}v_\mu^{(s)} + \gamma_5 a_\mu)q = \\ \bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_R + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v_\mu^{(s)}\right)q_L. \end{aligned}$$

2. Also verify

$$\bar{q}(s - i\gamma_5 p)q = \bar{q}_L(s - ip)q_R + \bar{q}_R(s + ip)q_L.$$

⇒ QCD Lagrangian with coupling to external fields

$$\mathcal{L} = \mathcal{L}_{\text{QCD}}^0 + \bar{q}_L \gamma^\mu \left( l_\mu + \frac{1}{3} v_\mu^{(s)} \right) q_L + \bar{q}_R \gamma^\mu \left( r_\mu + \frac{1}{3} v_\mu^{(s)} \right) q_R - \bar{q}_R (s + ip) q_L - \bar{q}_L (s - ip) q_R. \quad (*)$$

(\*) remains invariant under local transformations

$$q_R \mapsto \exp \left( -i \frac{\Theta(x)}{3} \right) V_R(x) q_R,$$
$$q_L \mapsto \exp \left( -i \frac{\Theta(x)}{3} \right) V_L(x) q_L,$$

$V_R(x)$  and  $V_L(x)$ : independent space-time-dependent SU(3) matrices, provided the external fields are subject to the transfor-

mations

$$\begin{aligned}r_\mu &\mapsto V_R r_\mu V_R^\dagger + iV_R \partial_\mu V_R^\dagger, \\l_\mu &\mapsto V_L l_\mu V_L^\dagger + iV_L \partial_\mu V_L^\dagger, \\v_\mu^{(s)} &\mapsto v_\mu^{(s)} - \partial_\mu \Theta, \\s + ip &\mapsto V_R (s + ip) V_L^\dagger, \\s - ip &\mapsto V_L (s - ip) V_R^\dagger.\end{aligned}$$

(Derivative terms in serve the same purpose as in the construction of gauge theories, i.e., they cancel analogous terms originating from the kinetic part of the quark Lagrangian.)

- **Practical implications of the local invariance**

**Allows one to also discuss a coupling to external gauge fields in the transition to the EFT.**

- 1. Coupling of the electromagnetic field to point-like fundamental particles results from gauging a U(1) symmetry. Here, the**



corresponding  $U(1)$  group is to be understood as a subgroup of a local  $SU(3)_L \times SU(3)_R$ .

2. Interaction of the light quarks with the charged and neutral gauge bosons of the weak interactions.

Q: What do we have to insert for the external fields to describe the **electromagnetic interaction** of quarks?

A:

$$r_\mu = l_\mu = -eQ\mathcal{A}_\mu, \quad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} : \text{ quark charge matrix}$$

Verification

$$\begin{aligned} \mathcal{L}_{\text{ext}} &= -e\mathcal{A}_\mu(\bar{q}_L Q \gamma^\mu q_L + \bar{q}_R Q \gamma^\mu q_R) = -e\mathcal{A}_\mu \bar{q} Q \gamma^\mu q \\ &= -e\mathcal{A}_\mu \left( \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d - \frac{1}{3} \bar{s} \gamma^\mu s \right) \\ &= -e\mathcal{A}_\mu J^\mu. \end{aligned}$$

“SU(2) version” of ChPT:

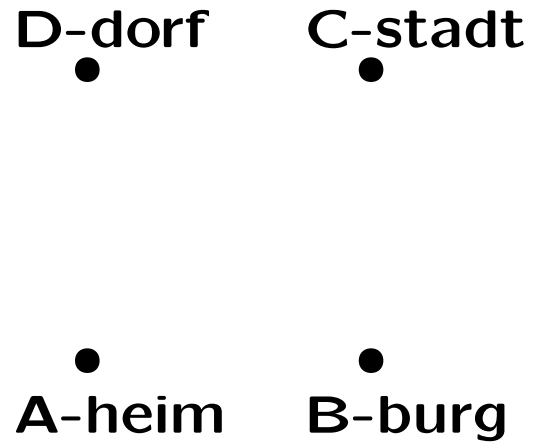
$$r_\mu = l_\mu = -e \frac{\tau_3}{2} \mathcal{A}_\mu, \quad v_\mu^{(s)} = -\frac{e}{2} \mathcal{A}_\mu,$$

because

$$Q = \frac{1}{6} \mathbf{1}_{2 \times 2} + \frac{\tau_3}{2}.$$

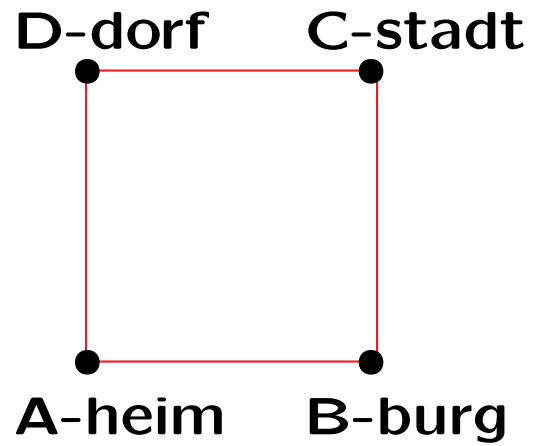
## Spontaneous Symmetry Breaking

Simple illustration



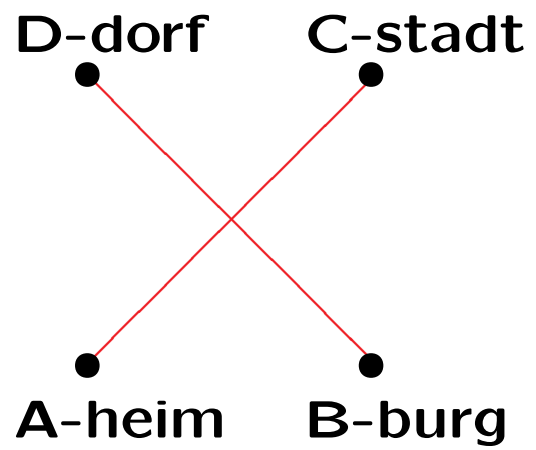
**Task: Find shortest road network connecting the four cities**

First guess



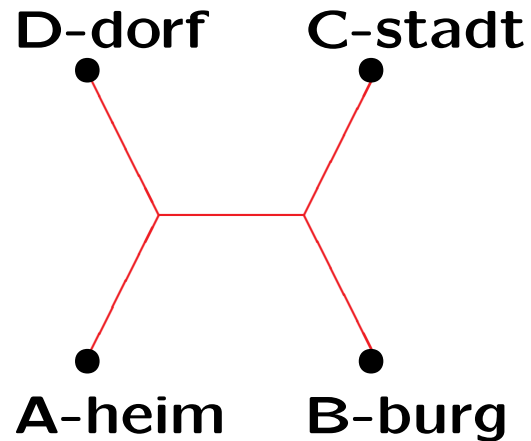
Total length:  $4a$  ( $a$  side length of square)

Second guess



Total length:  $2\sqrt{2}a < 4a$

## Final answer



Total length:  $(1 + \sqrt{3})a < 2\sqrt{2}a < 4a$

object	4 cities	Hamilton operator
symmetry	$D_4$	$G$
criterion	shortest road network	minimal energy
symmetry of solution	$D_2$	subgroup $H$ of $G$

## Example: O(3) sigma model

$$\begin{aligned}\mathcal{L}(\vec{\Phi}, \partial_\mu \vec{\Phi}) &= \mathcal{L}(\Phi_1, \Phi_2, \Phi_3, \partial_\mu \Phi_1, \partial_\mu \Phi_2, \partial_\mu \Phi_3) \\ &= \frac{1}{2} \partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2} \Phi_i \Phi_i - \frac{\lambda}{4} (\Phi_i \Phi_i)^2\end{aligned}$$

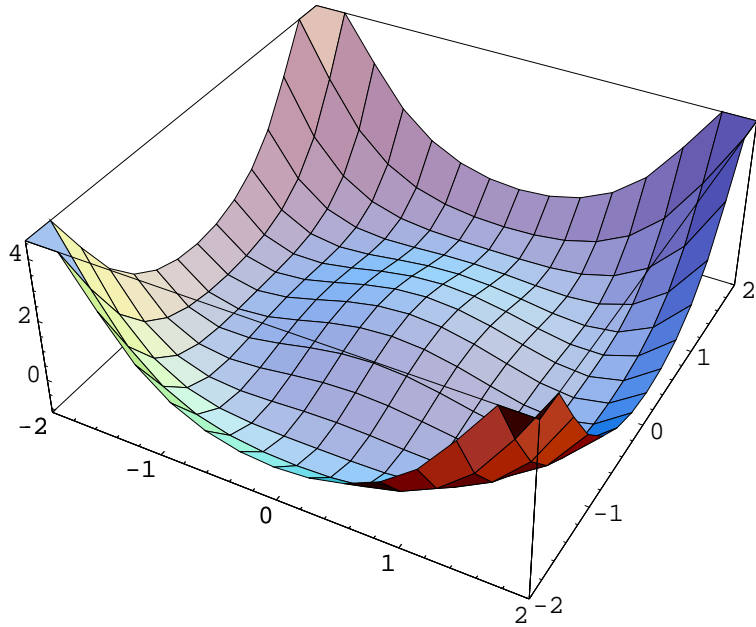
$m^2 < 0$ ,  $\lambda > 0$ , Hermitian fields  $\Phi_i$

$\mathcal{L}$  invariant under a global “isospin” rotation

$$g \in \mathbf{SO}(3) : \Phi_i \rightarrow \Phi'_i = D_{ij}(g) \Phi_j = (e^{-i\alpha_k T_k})_{ij} \Phi_j$$

$$[T_i, T_j] = i\epsilon_{ijk} T_k$$

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Two-dimensional rotationally invariant  
 potential:  $\mathcal{V}(x, y) = -(x^2 + y^2) + \frac{(x^2 + y^2)^2}{4}$

**Exercise:** Determine the minimum of the potential

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2} \Phi_i \Phi_i + \frac{\lambda}{4} (\Phi_i \Phi_i)^2$$

We find

$$|\vec{\Phi}_{\min}| = \sqrt{\frac{-m^2}{\lambda}} \equiv v, \quad |\vec{\Phi}| = \sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2}$$



$\vec{\Phi}_{\min}$  can point in any direction in isospin space  
 $\Rightarrow$  non-countably infinite number of degenerate vacua

Spontaneous symmetry breaking (hidden symmetry)

Select a particular direction which, by an appropriate orientation of the internal coordinate frame, we denote as the 3 direction,

$$\vec{\Phi}_{\min} = v\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$\vec{\Phi}_{\min}$  not invariant under full group  $G = \text{SO}(3)$

Rotations about the 1 and 2 axis change  $\vec{\Phi}_{\min}$

$$T_1\vec{\Phi}_{\min} = v \begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}, \quad T_2\vec{\Phi}_{\min} = v \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$$

$\vec{\Phi}_{\min}$  invariant under subgroup  $H$  of  $G$ : rotations about the 3 axis

$$h \in H : \quad \vec{\Phi}' = D(h)\vec{\Phi} = e^{-i\alpha_3 T_3}\vec{\Phi}, \quad D(h)\vec{\Phi}_{\min} = \vec{\Phi}_{\min}, \quad T_3\vec{\Phi}_{\min} = 0$$

**Exercise:** Expand  $\Phi_3$  with respect to  $v$ :  $\Phi_3(x) = v + \eta(x)$

New expression for the potential

$$\tilde{\mathcal{V}} = \frac{1}{2}(-2m^2)\eta^2 + \lambda v\eta(\Phi_1^2 + \Phi_2^2 + \eta^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 + \eta^2)^2 - \frac{\lambda}{4}v^4$$

$$m_{\Phi_1}^2 = m_{\Phi_2}^2 = 0, \quad m_{\eta}^2 = -2m^2$$

Model-independent feature of the above example:

For each of the two generators  $T_1$  and  $T_2$  which do not annihilate the ground state one obtains a massless Goldstone boson

Number of Goldstone bosons is determined by the structure of the symmetry groups:

- $G$  symmetry group of the Lagrangian,  $n_G$  generators
- $H$  subgroup with  $n_H$  generators which leaves the ground state after spontaneous symmetry breaking invariant
- # of Goldstone bosons:  $n_G - n_H$

## Explicit Symmetry Breaking: A First Look

Modify potential by adding  $a\Phi_3$ ,

$$\mathcal{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2 + a\Phi_3,$$

$m^2 < 0$ ,  $\lambda > 0$ ,  $a > 0$  and real fields  $\Phi_i$ .

New potential has **lower symmetry**:  $O(2)$  symmetry (rotations about the 3 axis)

Conditions for the new minimum (from  $\vec{\nabla}_{\Phi}\mathcal{V} = 0$ ) read

$$\Phi_1 = \Phi_2 = 0, \quad \lambda\Phi_3^3 + m^2\Phi_3 + a = 0$$

**Exercise:** Solve using a perturbative ansatz

$$\langle\Phi_3\rangle = \Phi_3^{(0)} + a\Phi_3^{(1)} + \mathcal{O}(a^2).$$

**Result**

$$\Phi_3^{(0)} = \pm\sqrt{-\frac{m^2}{\lambda}}, \quad \Phi_3^{(1)} = \frac{1}{2m^2}.$$

$\Phi_3^{(0)}$ : Result without explicit breaking.

Expand potential with  $\Phi_3 = \langle \Phi_3 \rangle + \chi \Rightarrow$

$$m_{\Phi_1}^2 = m_{\Phi_2}^2 = a \sqrt{\frac{\lambda}{-m^2}}, \quad \left( m_{\chi}^2 = -2m^2 + 3a \sqrt{\frac{\lambda}{-m^2}} \right).$$

Remarks:

- The Goldstone bosons have acquired a mass.
- Squared masses proportional to  $a$ .
- Quantum corrections lead to observables which are nonanalytic in the symmetry breaking parameter  $a$ , e.g.  $a \ln(a)$  (so-called chiral logarithms).
- Analogue of  $a$  in QCD: Quark masses

# Spontaneous Symmetry Breaking in QCD

## Indications from the Hadron Spectrum

Example:  $H_{\text{str}}$  is isospin invariant

$$[H_{\text{str}}, T_i] = 0, \quad [T_i, T_j] = i\epsilon_{ijk}T_k$$

Hadrons can be classified as irreducible multiplets of isospin SU(2)

$$\begin{aligned} T = 0 &: d \\ T = \frac{1}{2} &: \begin{pmatrix} p \\ n \end{pmatrix}, \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} \\ T = 1 &: \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \\ T = \frac{3}{2} &: \begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix} \end{aligned}$$

- Q: Where does this symmetry come from?
- A: Accidental global symmetry of QCD

Consider linear combinations

$$Q_V^a = Q_R^a + Q_L^a \xrightarrow{P} Q_V^a$$

$$Q_A^a = Q_R^a - Q_L^a \xrightarrow{P} -Q_A^a$$

**Exercise:** Commutation relations

$$[Q_V^a, Q_V^b] = if_{abc}Q_V^c, \quad [Q_V^a, Q_A^b] = if_{abc}Q_A^c, \quad [Q_A^a, Q_A^b] = if_{abc}Q_V^c$$

$$[H_{\text{QCD}}^0, Q_V^a] = [H_{\text{QCD}}^0, Q_A^a] = 0$$

Let

$$H_{\text{QCD}}^0|\Psi\rangle = E|\Psi\rangle, \quad P|\Psi\rangle = |\Psi\rangle$$

Construct new state  $|\Phi\rangle = Q_A|\Psi\rangle$  (superscript  $a$  suppressed)

$$H_{\text{QCD}}^0|\Phi\rangle = H_{\text{QCD}}^0Q_A|\Psi\rangle = Q_A \underbrace{H_{\text{QCD}}^0|\Psi\rangle}_{E|\Psi\rangle} = E|\Phi\rangle$$

$$P|\Phi\rangle = PQ_A|\Psi\rangle = \underbrace{PQ_AP^{-1}}_{-Q_A} \underbrace{P|\Psi\rangle}_{|\Psi\rangle} = -|\Phi\rangle$$

Not observed in hadronic spectrum

- Q: What's wrong?
- A: We have tacitly assumed that the ground state of QCD is annihilated by  $Q_A^a$ .



## Solution: Spontaneous symmetry breaking

Symmetry of  $|0\rangle \neq$  symmetry of  $H_{\text{QCD}}^0$

- Coleman theorem:<sup>4</sup> The symmetry of the ground state determines the symmetry of the spectrum (reverse argument: infer symmetry of the ground state from the symmetry of the spectrum)
- Goldstone theorem:<sup>5</sup> To each generator that does not annihilate the ground state exists a massless Goldstone boson

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<sup>4</sup>S. Coleman, J. Math. Phys. 7, 787 (1966)

<sup>5</sup>J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. 127, 965 (1962)

- Here

- $H_{\text{QCD}}^0$  invariant under  $G = \text{SU}(3)_L \times \text{SU}(3)_R$

- $|0\rangle$  invariant under

$$H = \{(V, V)\} \cong \text{SU}(3)_V \quad \text{flavor SU}(3)$$

- idealized: 8 massless Goldstone bosons  $\pi, K, \eta$

Another (sufficient but not necessary) criterion:<sup>6</sup> Nonvanishing scalar quark condensate in the chiral limit.

Analogy with a ferromagnet

$$-\langle \vec{M} \rangle \cdot \vec{H} \leftrightarrow \langle \bar{u}u \rangle_0 m_u + \langle \bar{d}d \rangle_0 m_d + \langle \bar{s}s \rangle_0 m_s$$

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<sup>6</sup>G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rev. Lett. 86, 5008 (2001)

## Chiral Perturbation Theory for Mesons

### Effective field theory<sup>7</sup>

... if one writes down the **most general possible Lagrangian**, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian **to any given order of perturbation theory**, the result will simply be the most general possible S–matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. ...

For our purposes:

Most general description of the strong interactions at low energies:  $\pi\pi$ ,  $\pi N$ ,  $NN$ , etc.

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<sup>7</sup>S. Weinberg, *Physica A* 96, 327 (1979)

Perturbative calculations in effective field theory require **two main ingredients**

- (1) Knowledge of the most general effective Lagrangian
- (2) Expansion scheme for observables in terms of a consistent power counting
  - (a) Tree-level diagrams, loop diagrams, regularization (of infinities)
  - (b) Renormalization
  - (c) Power counting scheme for renormalized diagrams

## Commonly used methods

1. Expansion in powers of coupling constants (e. g., QED)
2. Loop expansion (expansion in  $\hbar$ )
3. Momentum and quark mass expansion, ChPT

	Fundamental theory	Effective field theory
	QCD	ChPT
dof	quarks & gluons	Goldstone bosons (+ other hadrons)
parameters	$g_3$ + quark masses	( $\infty$ # of ) LECs + quark masses

## Simplified analogies between multipole expansion and EFT

Multipole expansion	EFT
$ \vec{x}  \gg R$	$q \ll \Lambda_\chi$
$\phi(\vec{x}) = \sum_{lm} \boxed{q_{lm}} \boxed{\frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}}$	$\mathcal{L}_{\text{eff}} = \sum_{lm} \boxed{c_{lm}} \boxed{\mathcal{L}_{lm}}$
multipole moment $q_{lm}$	LEC $c_{lm}$
$\frac{1}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$	Structures $\mathcal{L}_{lm}$

- In principle, infinite number of terms.  
Actual calculation: Truncation at finite order.
- Systematic improvement possible.

## Effective Lagrangian

Starting point of Chiral Perturbation Theory:<sup>8</sup>

Write down the most general Lagrangian in an expansion in (co-variant) derivatives ( $\rightarrow$  momenta) and quark masses ( $\rightarrow$  square of meson masses):

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$$

- Symmetry group of the Lagrangian as  $m_u, m_d, m_s \rightarrow 0$ :

$$\mathbf{SU}(3)_L \times \mathbf{SU}(3)_R \times \mathbf{U}(1)_V$$

- Symmetry group of the ground state:

$$H = \mathbf{SU}(3)_V \times \mathbf{U}(1)_V$$

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<sup>8</sup>J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* 158, 142 (1984);  
*Nucl. Phys. B*250, 465 (1985)



- 8 pseudoscalar dynamical degrees of freedom which transform as an octet with respect to  $SU(3)_V$
- Include explicit chiral symmetry breaking through quark masses as a perturbation

## Construction of the lowest-order Lagrangian

Following Gasser and Leutwyler the consequences of the  $SU(3)_L \times SU(3)_R \times U(1)_V$  symmetry of  $\mathcal{L}_{\text{QCD}}^0$  are analyzed by

- introducing a coupling to color-neutral, Hermitian external fields:

$$\begin{aligned}
 \mathcal{L} &= \mathcal{L}_{\text{QCD}}^0 + \mathcal{L}_{\text{ext}} \\
 &= \mathcal{L}_{\text{QCD}}^0 + \bar{q} \gamma_\mu (v^\mu + \frac{1}{3} v_{(s)}^\mu + \gamma_5 a^\mu) q - \bar{q} (s - i \gamma_5 p) q
 \end{aligned}$$

where

$$v^\mu = \sum_{a=1}^8 \frac{\lambda_a}{2} v_a^\mu, \quad a^\mu = \sum_{a=1}^8 \frac{\lambda_a}{2} a_a^\mu,$$

$$s = \sum_{a=0}^8 \lambda_a s_a, \quad p = \sum_{a=0}^8 \lambda_a p_a, \quad \lambda_0 = \sqrt{2/3} \text{diag}(1, 1, 1)$$

Ordinary three flavor QCD Lagrangian:

$$v^\mu = v_{(s)}^\mu = a^\mu = p = 0, \quad s = \text{diag}(m_u, m_d, m_s)$$

Introduce

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu$$

**Exercise**

Using the projection operators  $P_L$  and  $P_R$ , verify

$$\bar{q} \gamma^\mu (v_\mu + \frac{1}{3} v_\mu^{(s)} + \gamma_5 a_\mu) q = \bar{q}_R \gamma^\mu \left( r_\mu + \frac{1}{3} v_\mu^{(s)} \right) q_R + \bar{q}_L \gamma^\mu \left( l_\mu + \frac{1}{3} v_\mu^{(s)} \right) q_L$$

$$\bar{q} (s - i \gamma_5 p) q = \bar{q}_L (s - i p) q_R + \bar{q}_R (s + i p) q_L$$

- and promoting the global symmetry to a local symmetry:

$$L \rightarrow V_L(x), R \rightarrow V_R(x)$$

## Building blocks of the effective Lagrangian

Collect Goldstone boson fields in Hermitian, traceless,  $3 \times 3$  matrix

$$\phi(x) = \sum_{a=1}^8 \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

with the Gell-Mann matrices  $\lambda_a$  and  $\phi_a(x) = \frac{1}{2} \text{Tr}[\lambda_a \phi(x)]$

Define special unitary matrix

$$U(x) = \exp\left(i \frac{\phi(x)}{F_0}\right)$$

Transformation behavior under  $G = \text{SU}(3)_L \times \text{SU}(3)_R$ , parity  $P$ , and charge conjugation  $C$ :

$$\begin{aligned}
 U &\xrightarrow{G} V_R U V_L^\dagger \quad * \\
 U(\vec{x}, t) &\xrightarrow{P} U^\dagger(-\vec{x}, t) \\
 U &\xrightarrow{C} U^T
 \end{aligned}$$

Introduce a covariant derivative and field strength tensors

$$\begin{aligned}
 D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu \xrightarrow{G} V_R D_\mu U V_L^\dagger \\
 f_{\mu\nu}^R &\equiv \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] \xrightarrow{G} V_R f_{\mu\nu}^R V_R^\dagger \\
 f_{\mu\nu}^L &\equiv \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu] \xrightarrow{G} V_L f_{\mu\nu}^L V_L^\dagger
 \end{aligned}$$

and the linear combination  $\chi = 2B_0(s + ip)$

E.g., pure QCD:  $\chi = 2B_0 \text{diag}(m_u, m_d, m_s)$

Construct the effective Lagrangian in terms of  $U$ ,  $U^\dagger$ ,  $\chi$ ,  $\chi^\dagger$ ,  $f_{\mu\nu}^R$ ,  $f_{\mu\nu}^L$  and covariant derivatives of these objects.

---

\* Remark:

Consider elements  $(V_L, V_R)$  of  $G = \text{SU}(3)_L \times \text{SU}(3)_R$  of the type

- $(V, V) : (V_1, V_1)(V_2, V_2) = (V_1V_2, V_1V_2)$   
form subgroup  $H$  of  $G$
- $(A^\dagger, A) : (A_1^\dagger, A_1)(A_2^\dagger, A_2) \neq ((A_1A_2)^\dagger, A_1A_2)$   
no subgroup

If we define the ground state as :  $\phi_a(x) = 0$  or  $U_0 = 1_{3 \times 3}$

- $U_0$  invariant with respect to  $(V, V)$ :  
 $U_0 \rightarrow VU_0V^\dagger = VV^\dagger U_0 = U_0$

- $U_0$  not invariant with respect to  $(A^\dagger, A)$ :

$$U_0 \rightarrow AU_0A = A^2U_0 \neq U_0$$


---

## Construction of invariants

Suppose we have matrices  $A, B, C, \dots$ , all of which transform as

$$A \xrightarrow{G} V_R A V_L^\dagger$$

$$B \xrightarrow{G} V_R B V_L^\dagger$$

...

Form invariants by “multiplying” in the following way:

$$\begin{aligned} \text{Tr}(AB^\dagger) &\xrightarrow{G} \text{Tr}(V_R A \underbrace{V_L^\dagger V_L}_1 B^\dagger V_R^\dagger) \\ &= \text{Tr}(V_R^\dagger V_R AB^\dagger) = \text{Tr}(AB^\dagger) \end{aligned}$$

- Generalization to more terms is obvious
- Product of invariant traces is invariant
- Assign (chiral) orders:

$$\begin{aligned}
 U &= \mathcal{O}(q^0) \\
 D_\mu U &= \mathcal{O}(q) \\
 r_\mu, l_\mu &= \mathcal{O}(q) \\
 f_{\mu\nu}^{L/R} &= \mathcal{O}(q^2) \\
 \chi &= \mathcal{O}(q^2)
 \end{aligned}$$

- List of objects  $A$  up to and including order  $q^2$  which transform as  $A' = V_R A V_L^\dagger$ :

$$U, D_\mu U, D_\mu D_\nu U, \chi, U f_{\mu\nu}^L, f_{\mu\nu}^R U$$

- Construction of chirally invariant expressions (to order  $q^2$ ):

$$\mathcal{O}(q^0) : \text{Tr} \left( UU^\dagger \right) = \text{Tr}(1) = \text{const.}$$

$$\mathcal{O}(q) : \text{Tr} \left( D_\mu UU^\dagger \right) = 0$$

**important: excludes terms of the type  $\text{Tr}[\mathcal{O}(q)] \times \text{Tr}(\dots)$**

$$\begin{aligned} \mathcal{O}(q^2) : & \text{Tr} \left( D_\mu D_\nu UU^\dagger \right) \left( = -\text{Tr} \left[ D_\nu U (D_\mu U)^\dagger \right] \right) \\ & \text{Tr} \left[ D_\mu U (D_\nu U)^\dagger \right] \\ & \text{Tr} \left[ U (D_\mu D_\nu U)^\dagger \right] \left( = -\text{Tr} \left[ D_\mu U (D_\nu U)^\dagger \right] \right) \\ & \text{Tr} \left( \chi U^\dagger \right) \\ & \text{Tr} \left( U \chi^\dagger \right) \\ & \text{Tr} \left[ (U f_{\mu\nu}^L) U^\dagger \right] = \text{Tr} \left( f_{\mu\nu}^L \right) \\ & \text{Tr} \left( f_{\mu\nu}^R \right) \end{aligned}$$



- Lorentz invariance:

Indices have to be contracted

$$g^{\mu\nu} f_{\mu\nu}^L = g^{\mu\nu} f_{\mu\nu}^R = 0$$

- Candidates:

$$\begin{aligned} & \text{Tr} \left[ D_\mu U (D^\mu U)^\dagger \right] \\ & \text{Tr} \left( \chi U^\dagger \pm U \chi^\dagger \right) \end{aligned}$$

- Parity:

$$\mathcal{L}(\vec{x}, t) \xrightarrow{P} \mathcal{L}(-\vec{x}, t)$$

$\text{Tr}(\chi U^\dagger - U \chi^\dagger)$  has wrong parity

- Charge conjugation [no additional constraint at  $\mathcal{O}(q^2)$ ]

## Lowest-order Lagrangian $\mathcal{L}_2$

$$\mathcal{L}_2 = \frac{F_0^2}{4} \text{Tr} \left[ D_\mu U (D^\mu U)^\dagger \right] + \frac{F_0^2}{4} \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right)$$

At  $\mathcal{O}(q^2)$  two parameters:

$$F_0 \approx 93 \text{ MeV}, \quad 3F_0^2 B_0 = -\langle 0 | \bar{q}q | 0 \rangle$$

- $\mathcal{L}_2$  has predictive power!
- However, we first need to discuss the power counting scheme.

## Weinberg's power counting for the mesonic sector<sup>9</sup>

Q: How do different diagrams compare?

$$\mathcal{M}(tp_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q) = \mathcal{O}(q^D)$$

For small enough momenta (and masses) contributions with increasing  $D$  become less important

$$\begin{aligned} D &= nN_L - 2N_I + \sum_{k=1}^{\infty} 2kN_{2k} \\ &= 2 + (n - 2)N_L + \sum_{k=1}^{\infty} 2(k - 1)N_{2k} \\ &\geq 2 \text{ in 4 dimensions} \end{aligned}$$

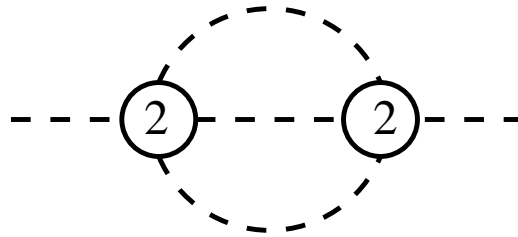
- $N_L$ : Number of independent loops

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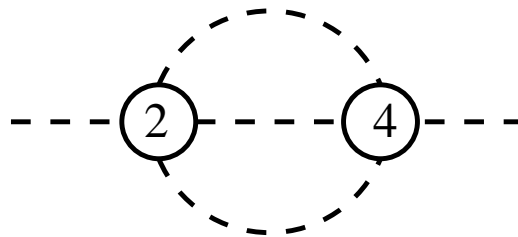
<sup>9</sup>S. Weinberg, *Physica A* 96, 327 (1979)

- $N_I$ : Number of internal Goldstone boson lines
- $N_{2k}$ : Number of vertices from  $\mathcal{L}_{2k}$
- Loops suppressed by  $(n - 2)N_L$
- Relation between the momentum and loop expansion
- Perturbative scheme in terms of **external momenta** and **quark masses** ( $\rightarrow$  meson masses<sup>2</sup>) which are small compared to some scale [here:  $4\pi F_0 = \mathcal{O}(1\text{GeV})$ ]

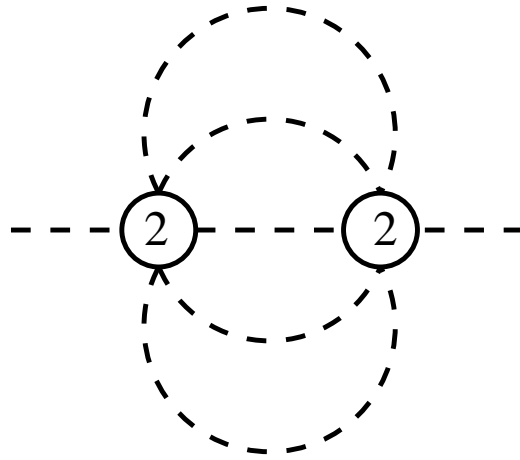
## Examples ( $n = 4$ dimensions)



$$\begin{aligned} D &= 4 \cdot 2 - 2 \cdot 3 + 2 \cdot 2 = 6 \\ &= 2 + 2 \cdot 2 + (2 - 2) \cdot 2 \end{aligned}$$



$$D = 4 \cdot 2 - 2 \cdot 3 + 1 \cdot 2 + 1 \cdot 4 = 8$$



$$D = 4 \cdot 4 - 2 \cdot 5 + 2 \cdot 2 = 10$$

**Proof:**

$N_I$ : # of internal Goldstone boson lines

$N_{2k}$ : # of vertices with  $2k$  derivatives or  $k$  quark mass terms

● **Internal lines:**

$$\int d^4k \frac{1}{k^2 - M^2 + i\epsilon} \quad \begin{array}{c} M^2 \rightarrow t^2 M^2 \\ \rightarrow \\ k \equiv tl \end{array} \quad \int d^4k \frac{1}{t^2(k^2/t^2 - M^2 + i\epsilon)}$$
$$t^2 \int d^4l \frac{1}{l^2 - M^2 + i\epsilon}$$

● **Vertices with  $2k$  derivatives or  $k$  quark mass terms:**

$$\delta^4(q) q^{2k} \rightarrow t^{2k-4} \delta^4(q) q^{2k}$$

– since  $p \rightarrow tp$  if  $q$  is an external momentum

– and  $k = tl$  if  $q$  is an internal momentum (see above)

- These are the rules to calculate  $S \sim \delta^4(p)\mathcal{M}$   
add 4 to compensate for the overall delta function

- Scaling behavior of the contribution to  $\mathcal{M}$  of a given diagram

$$D = 4 + 2N_I + \sum_{k=1}^{\infty} N_{2k}(2k - 4)$$

- Relation between # of loops  $N_L$ , # of vertices  $N_V$ , and # of internal lines:

$$N_I = N_L + N_V - 1 = N_L + \sum_{k=1}^{\infty} N_{2k} - 1$$



•

$$D = 2 + \sum_{k=1}^{\infty} (2k - 2)N_{2k} + 2N_L \geq 2$$

**In particular, diagrams containing loops are suppressed due to the term  $2N_L$**

## Simple applications at lowest order

### Goldstone boson masses due to quark masses

No external sources:  $D_\mu U \rightarrow \partial_\mu U$ ,  $\chi = 2B_0\mathcal{M}$ ,  $m_u = m_d = \hat{m}$

$$\mathcal{L}_2 \stackrel{\text{Exercise}}{=} \underbrace{\frac{1}{2} \left( \partial_\mu \pi^0 \partial^\mu \pi^0 - M_\pi^2 \pi^0 \pi^0 \right) + \dots}_{\text{sum of free Lagrangians}} + \underbrace{\mathcal{L}_{\text{int}}}_{O(\phi^4)}$$

Read off

$$\begin{aligned} M_\pi^2 &= 2B_0\hat{m} \\ M_K^2 &= B_0(\hat{m} + m_s) \\ M_\eta^2 &= \frac{2}{3}B_0(\hat{m} + 2m_s) \end{aligned}$$

**Remark:** Without additional information about  $B_0$  one cannot determine the absolute values of the quark masses.

$$\frac{M_K^2}{M_\pi^2} = \frac{\hat{m} + m_s}{2\hat{m}} \Rightarrow \frac{m_s}{\hat{m}} = 25.9$$

$$\frac{M_\eta^2}{M_\pi^2} = \frac{2m_s + \hat{m}}{3\hat{m}} \Rightarrow \frac{m_s}{\hat{m}} = 24.3$$

**Gell-Mann-Okubo formula**

$$4M_K^2 = 3M_\eta^2 + M_\pi^2$$

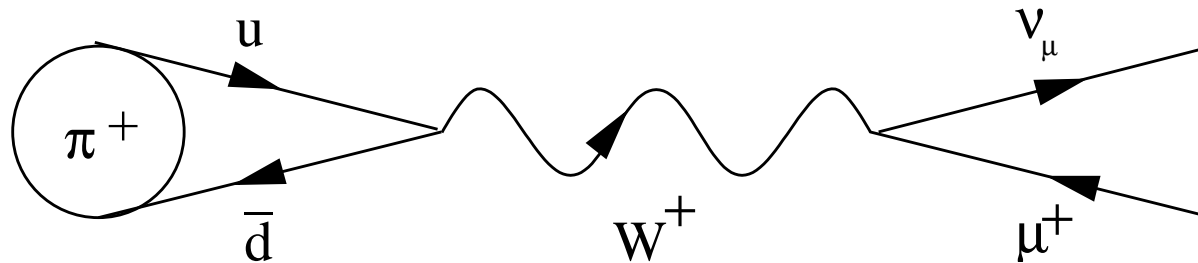
**Insert**

$$M_K = 496 \text{ MeV}, \quad M_\pi = 135 \text{ MeV}$$

**and “predict”**

$$M_\eta = 567 \text{ MeV}, \quad \text{experimental value: } 547 \text{ MeV}$$

## Pion decay $\pi^+ \rightarrow \mu^+ \nu_\mu$



- Interaction of quarks with the massive charged weak bosons

$$\mathcal{W}_\mu^\pm = (\mathcal{W}_{1\mu} \mp i\mathcal{W}_{2\mu})/\sqrt{2}$$

$$\mathcal{L}_{\text{CC}}^{(q)} = -\frac{g}{2\sqrt{2}} \left\{ \mathcal{W}_\mu^+ [V_{ud}\bar{u}\gamma^\mu(1-\gamma_5)d + V_{us}\bar{u}\gamma^\mu(1-\gamma_5)s] + h.c. \right\}$$

$$|V_{ud}| = 0.97425 \pm 0.00022, \quad |V_{us}| = 0.2252 \pm 0.0009$$

Fermi constant is related to the gauge coupling  $g$  and the  $W$  mass as

$$G_F = \sqrt{2} \frac{g^2}{8M_W^2} = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}$$

Set

$$r_\mu = 0, \quad l_\mu = -\frac{g}{\sqrt{2}}(\mathcal{W}_\mu^+ T_+ + h.c.)$$

in  $\mathcal{L}_{\text{ext}}$ , where

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Coupling of the  $W$  bosons to the leptons

$$\mathcal{L}_{\text{CC}}^{(l)} = -\frac{g}{2\sqrt{2}} \left[ \mathcal{W}_\mu^+ \bar{\nu}_\mu \gamma^\mu (1 - \gamma_5) \mu + \mathcal{W}_\mu^- \bar{\mu} \gamma^\mu (1 - \gamma_5) \nu_\mu \right]$$

- Coupling of the  $W$  bosons to Goldstone bosons

Insert covariant derivative

$$D_\mu U = \partial_\mu U + iU l_\mu$$

into  $\mathcal{L}_2$

$$\begin{aligned}\frac{F_0^2}{4} \text{Tr}[D_\mu U (D^\mu U)^\dagger] &= i \frac{F_0^2}{2} \text{Tr}(l_\mu \partial^\mu U^\dagger U) + \dots = \frac{F_0}{2} \text{Tr}(l_\mu \partial^\mu \Phi) + \dots \\ &= -g \frac{F_0}{2} \left[ \mathcal{W}_\mu^+ (V_{ud} \partial^\mu \pi^- + V_{us} \partial^\mu K^-) \right. \\ &\quad \left. + \mathcal{W}_\mu^- (V_{ud} \partial^\mu \pi^+ + V_{us} \partial^\mu K^+) \right]\end{aligned}$$

- Feynman propagator for  $W$  bosons

$$\frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}}{k^2 - M_W^2} = \frac{g_{\mu\nu}}{M_W^2} + O\left(\frac{k k}{M_W^4}\right)$$

- Feynman rule for the invariant amplitude for the weak pion

decay

$$\mathcal{M} = \underbrace{i \left[ -\frac{g}{2\sqrt{2}} \bar{u}_{\nu\mu} \gamma^\rho (1 - \gamma_5) v_{\mu^+} \right]}_{\text{leptonic vertex}} \underbrace{\frac{ig_{\rho\sigma}}{M_W^2}}_{W \text{ propagator}} \underbrace{i \left[ -g \frac{F_0}{2} V_{ud} (-ip^\sigma) \right]}_{\text{hadronic vertex}}$$

$$= -G_F V_{ud} F_0 \bar{u}_{\nu\mu} \not{p} (1 - \gamma_5) v_{\mu^+}$$

$p$ : four-momentum of the pion

- Decay rate

$$\frac{1}{\tau} = \frac{G_F^2 |V_{ud}|^2}{4\pi} F_0^2 M_\pi m_\mu^2 \left( 1 - \frac{m_\mu^2}{M_\pi^2} \right)^2$$

- $F_0$ : pion-decay constant in the chiral limit

- Empirical numbers:

$$\begin{aligned} F_\pi &= 92.2 \text{ MeV} \\ F_K &= 113 \text{ MeV} \end{aligned}$$

$\pi\pi$  scattering from  $\mathcal{L}_2^{4\phi}$

Consider the Lagrangian

$$\mathcal{L}_2 = \frac{F^2}{4} \text{Tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{F^2}{4} \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right)$$

in the SU(2) sector with

$$\chi = 2B \underbrace{\begin{pmatrix} \hat{m} & 0 \\ 0 & \hat{m} \end{pmatrix}}_{\mathcal{M}}$$

and

$$U = \exp \left( i \frac{\phi}{F} \right), \quad \phi = \sum_{i=1}^3 \tau_i \phi_i =: \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.$$

Remark on **chiral limit**:

- In the SU(2) sector it is common to express quantities in the chiral limit without index 0, e. g.,  $F$  and  $B$ . By this one means



the SU(2) chiral limit, i. e.  $m_u = m_d = 0$  but  $m_s$  at its physical value.

- In the SU(3) sector the quantities  $F_0$  and  $B_0$  denote the chiral limit for all three quarks:  $m_u = m_d = m_s = 0$ .

Substitution  $U \leftrightarrow U^\dagger$ .  $\Rightarrow \mathcal{L}_2$  contains even powers of  $\phi$  only:

$$\mathcal{L}_2 = \mathcal{L}_2^{2\phi} + \mathcal{L}_2^{4\phi} + \dots$$

- $\mathcal{L}_2$  does not produce a vertex with 3 Goldstone bosons.  $\Rightarrow$  At  $D = 2$ , no  $s$ -,  $u$ -, and  $t$ -channel pole diagrams.
- At  $D = 2$ ,  $\pi\pi$  scattering is generated by a 4 Goldstone boson interaction term.

Expand

$$U = 1 + i\frac{\phi}{F} - \frac{1}{2}\frac{\phi^2}{F^2} - \frac{i}{6}\frac{\phi^3}{F^3} + \frac{1}{24}\frac{\phi^4}{F^4} + \dots$$

and identify  $\mathcal{L}_2^{4\phi}$  as (**Exercise**)

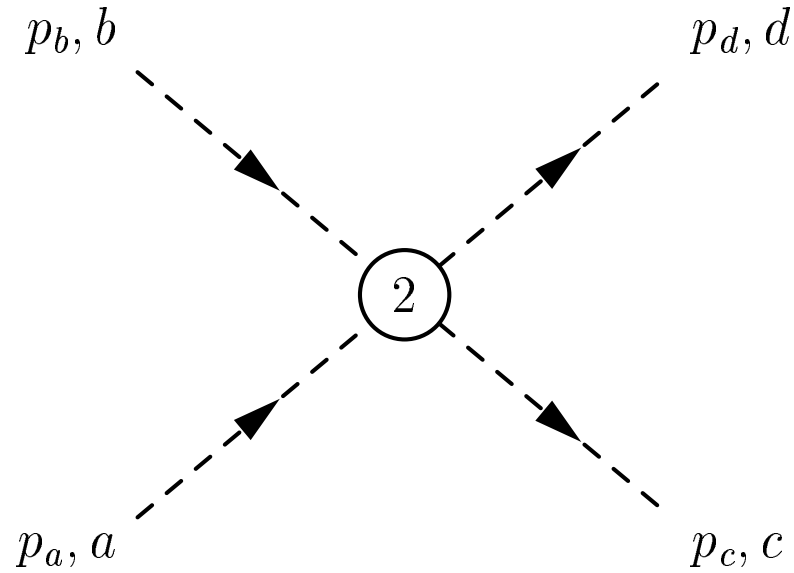
$$\mathcal{L}_2^{4\phi} = \frac{1}{48F^2} \left[ \text{Tr}([\phi, \partial_\mu \phi][\phi, \partial^\mu \phi]) + 2B \text{Tr}(\mathcal{M}\phi^4) \right].$$

**Remark:** Substituting  $F \rightarrow F_0$ ,  $B \rightarrow B_0$  and the relevant expressions for  $\phi$  and the quark mass matrix  $\mathcal{M}$  the corresponding formula for SU(3) looks identical.

Insert  $\phi = \tau_i \phi_i \Rightarrow$  (**Exercise**)

$$\begin{aligned} \mathcal{L}_2^{4\phi} &= -\frac{1}{6F^2} \epsilon_{ijm} \phi_i \partial_\mu \phi_j \epsilon_{klm} \phi_k \partial^\mu \phi_l + \frac{M^2}{24F^2} \phi_i \phi_i \phi_j \phi_j \\ &= \frac{1}{6F^2} (\phi_i \partial^\mu \phi_i \partial_\mu \phi_j \phi_j - \phi_i \phi_i \partial_\mu \phi_j \partial^\mu \phi_j) + \frac{M^2}{24F^2} \phi_i \phi_i \phi_j \phi_j, \end{aligned}$$

where  $M^2 = 2B\hat{m}$ .



**Feynman rule for Cartesian isospin indices  $a, b, c,$  and  $d$ :**

$$\mathcal{M} = i \left[ \delta^{ab} \delta^{cd} \frac{s - M^2}{F^2} + \delta^{ac} \delta^{bd} \frac{t - M^2}{F^2} + \delta^{ad} \delta^{bc} \frac{u - M^2}{F^2} \right] - \frac{i}{3F^2} \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) (\Lambda_a + \Lambda_b + \Lambda_c + \Lambda_d),$$

where  $\Lambda_k = p_k^2 - M^2$ .

## Mandelstam variables

$$s = (p_a + p_b)^2 = (p_c + p_d)^2,$$

$$t = (p_a - p_c)^2 = (p_d - p_b)^2,$$

$$u = (p_a - p_d)^2 = (p_c - p_b)^2$$

and

$$2p_a \cdot p_b = s - p_a^2 - p_b^2, \quad 2p_c \cdot p_d = s - p_c^2 - p_d^2,$$

$$-2p_a \cdot p_c = t - p_a^2 - p_c^2, \quad -2p_b \cdot p_d = t - p_b^2 - p_d^2,$$

$$-2p_a \cdot p_d = u - p_a^2 - p_d^2, \quad -2p_b \cdot p_c = u - p_b^2 - p_c^2.$$

The last line of the Feynman rule disappears, if the external lines satisfy mass shell conditions.

Scattering process  $\pi^a(p_a) + \pi^b(p_b) \rightarrow \pi^c(p_c) + \pi^d(p_d)$  at  $\mathcal{O}(q^2)$ :

$$T = \delta^{ab}\delta^{cd}\frac{s - M_\pi^2}{F_\pi^2} + \delta^{ac}\delta^{bd}\frac{t - M_\pi^2}{F_\pi^2} + \delta^{ad}\delta^{bc}\frac{u - M_\pi^2}{F_\pi^2}$$

We replaced

$$F \rightarrow F_\pi, \quad F_\pi = F(1 + \mathcal{O}(q^2)),$$
$$M^2 \rightarrow M_\pi^2, \quad M_\pi^2 = M^2(1 + \mathcal{O}(q^2)),$$

because the difference is of  $\mathcal{O}(q^4)$  in  $T$ .

Consider (theoretical) limit  $M_\pi^2, s, t, u \rightarrow 0$ :

$$T \rightarrow 0$$

- Goldstone bosons interact “weakly” at low energies.

**Isospin symmetry.**  $\Rightarrow$  Most general parametrization

$$T = \delta^{ab}\delta^{cd}A(s, t, u) + \delta^{ac}\delta^{bd}A(t, s, u) + \delta^{ad}\delta^{bc}A(u, t, s)$$

with  $A(s, t, u) = A(s, u, t)$ .

**Isospin channels:**

$$T^{I=0} = 3A(s, t, u) + A(t, u, s) + A(u, s, t)$$

$$T^{I=1} = A(t, u, s) - A(u, s, t)$$

$$T^{I=2} = A(t, u, s) + A(u, s, t)$$

**s-wave scattering lengths**

$$T^{I=0}|_{\text{thr}} = 32\pi a_0^0$$

$$T^{I=2}|_{\text{thr}} = 32\pi a_0^2$$

- $\pi^+\pi^+$  scattering described by  $T^{I=2}$ .
- Other physical reactions may be determined using the appropriate Clebsch-Gordan coefficients.

Evaluate  $T$  matrices at threshold.  $\Rightarrow$   $s$ -wave  $\pi\pi$  scattering lengths<sup>10</sup>

$$T^{I=0}|_{\text{thr}} = 32\pi a_0^0, \quad T^{I=2}|_{\text{thr}} = 32\pi a_0^2.$$

Lower index 0:  $s$  wave; upper index: Isospin.

( $T^{I=1}|_{\text{thr}}$  vanishes because of Bose symmetry.)

Prediction at  $\mathcal{O}(q^2)$ :

$$A(s, t, u) = \frac{s - M_\pi^2}{F_\pi^2}.$$

At threshold

$$s_{\text{thr}} = (2M_\pi)^2$$

and thus

$$A(s_{\text{thr}}, t_{\text{thr}}, u_{\text{thr}}) = \frac{3M_\pi^2}{F_\pi^2}.$$

---

<sup>10</sup>The convention in ChPT differs by a factor  $(-M_\pi)$  from the usual definition of a scattering length in the effective range expansion.

- $I = 0$ : Consider linear combination

$$\begin{aligned}
& [3A(s, t, u) + A(t, u, s) + A(u, s, t)]_{\text{thr}} \\
&= [2A(s, t, u) + A(s, t, u) + A(t, u, s) + A(u, s, t)]_{\text{thr}} \\
&= \frac{6M_\pi^2}{F_\pi^2} + \frac{[s + t + u - 3M_\pi^2]_{\text{thr}}}{F_\pi^2} \\
&= \frac{7M_\pi^2}{F_\pi^2}
\end{aligned}$$

- $I = 2$ : Consider linear combination

$$\begin{aligned}
& [A(t, u, s) + A(u, s, t)]_{\text{thr}} \\
&= [A(t, u, s) + A(u, s, t) + A(s, t, u) - A(s, t, u)]_{\text{thr}} \\
&= \frac{M_\pi^2}{F_\pi^2} - \frac{3M_\pi^2}{F_\pi^2} \\
&= -\frac{2M_\pi^2}{F_\pi^2}.
\end{aligned}$$



- $\Rightarrow$  Famous results of current algebra for the scattering lengths:<sup>11</sup>

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.156, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045.$$

( $F_\pi = 93.2$  MeV and  $M_\pi = 139.57$  MeV)

- **Absolute prediction** of chiral symmetry! Once we know  $F_\pi$  (from pion decay) we can **predict** the scattering lengths.
- Different from Wigner-Eckart theorem which predicts relations among processes of the same type.

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<sup>11</sup>S. Weinberg, Phys. Rev. Lett. 17, 616 (1966)

## Experimental data

$$\pi^\pm p \rightarrow \pi^\pm \pi^+ n:^{12} a_0^0 = 0.204 \pm 0.014 \text{ (stat)} \pm 0.008 \text{ (syst)},$$

$$K^+ \rightarrow \pi^+ \pi^- e^+ \nu_e:^{13} a_0^0 = 0.216 \pm 0.013 \text{ (stat)} \pm 0.002 \text{ (syst)} \\ \pm 0.002 \text{ (theor)},$$

$$\pi^+ \pi^- \text{ atom lifetime:}^{14} |a_0^0 - a_0^2| = 0.264_{-0.020}^{+0.033},$$

$$K^\pm \rightarrow \pi^\pm \pi^0 \pi^0:^{15} a_0^0 - a_0^2 = 0.268 \pm 0.010 \text{ (stat)} \pm 0.004 \text{ (syst)} \\ \pm 0.013 \text{ (ext)},$$

$$a_0^2 = -0.041 \pm 0.022 \text{ (stat)} \pm 0.014 \text{ (syst)}.$$

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<sup>12</sup>M. Kermani et al. [CHAOS Collaboration], Phys. Rev. C 58, 3431 (1998)

<sup>13</sup>S. Pislak et al., Phys. Rev. D 67, 072004 (2003)

<sup>14</sup>B. Adeva et al. [DIRAC Collaboration], Phys. Lett. B 619, 50 (2005)

<sup>15</sup>J. R. Batley et al. [NA48/2 Collaboration], Phys. Lett. B 633, 173 (2006)

Predictions for the  $s$ -wave scattering lengths at  $\mathcal{O}(q^6)$ <sup>16</sup>

$$a_0^0 = \underbrace{\mathcal{O}(q^2)}_{\overline{0.156}} + \underbrace{\mathcal{O}(q^4): +28\%}_{\overline{0.039} + \overline{0.005}}_{\text{L anal.}} + \underbrace{\mathcal{O}(q^6): +8.5\%}_{\overline{0.013} + \overline{0.003} + \overline{0.001}}_{k_i \text{ L anal.}} = \overline{0.217},$$

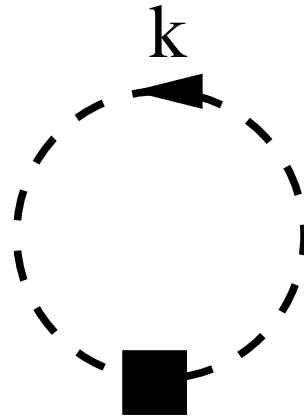
$$a_0^0 - a_0^2 = \underbrace{\mathcal{O}(q^2)}_{\overline{0.201}} + \underbrace{\mathcal{O}(q^4): +21\%}_{\overline{0.036} + \overline{0.006}}_{\text{L anal.}} + \underbrace{\mathcal{O}(q^6): +6.6\%}_{\overline{0.012} + \overline{0.003} + \overline{0.001}}_{k_i \text{ L anal.}} = \overline{0.258}.$$

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<sup>16</sup>J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, Phys. Lett. B 374, 210 (1996)

## Dimensional regularization: Basics

### Simple example



$$I = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i0^+}, \quad k^2 = k_0^2 - \vec{k}^2$$

Introduce

$$a \equiv \sqrt{\vec{k}^2 + M^2} > 0$$

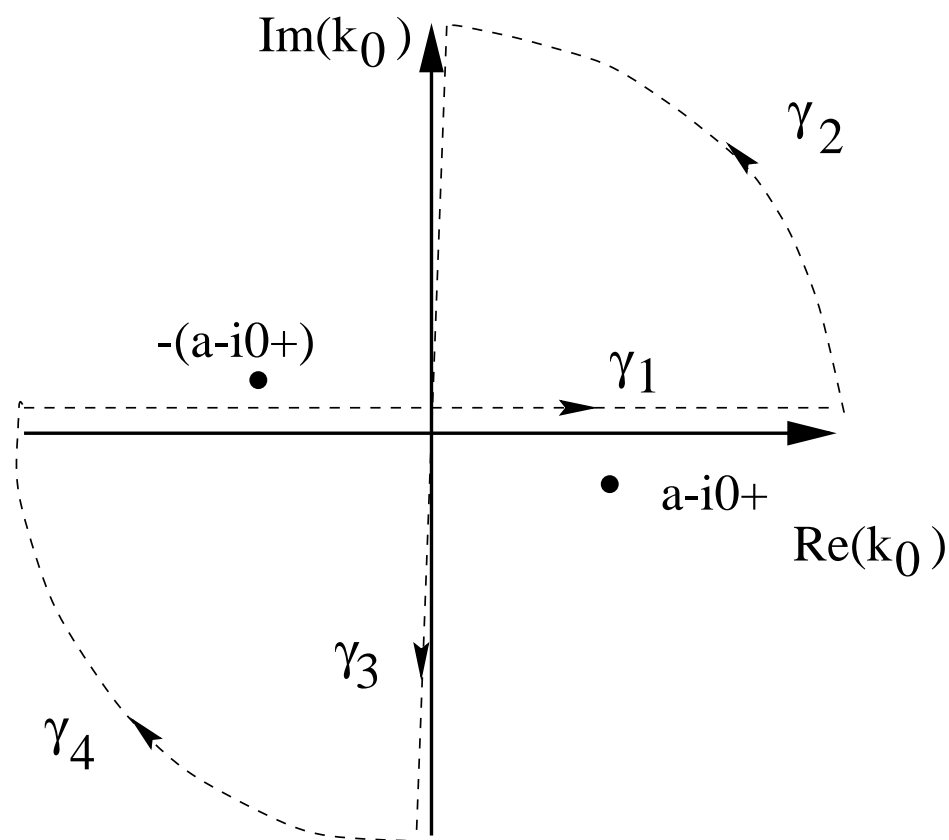
and define

$$f(k_0) = \frac{1}{[k_0 + (a - i0^+)][k_0 - (a - i0^)]}$$

Consider  $f$  in the complex  $k_0$  plane and make use of Cauchy's theorem

$$\oint_C dz f(z) = 0$$

for functions which are differentiable in every point inside the closed contour  $C$



$$0 = \sum_{i=1}^4 \int_{\gamma_i} dz f(z)$$

$$\int_{\gamma_1} f(z) dz = \int_{-\infty}^{\infty} f(t) dt$$

$$\int_{\gamma_2} f(z) dz = 0$$

$$\int_{\gamma_3} f(z) dz = \int_{\infty}^{-\infty} f(it) i dt$$

$$\int_{\gamma_4} f(z) dz = 0$$

$\Rightarrow$  the so-called Wick rotation

$$\int_{-\infty}^{\infty} f(t) dt = -i \int_{\infty}^{-\infty} dt f(it) = i \int_{-\infty}^{\infty} dt f(it)$$

## Intermediate result

$$\begin{aligned} I &= \frac{1}{(2\pi)^4} i \int_{-\infty}^{\infty} dk_0 \int d^3k \frac{i}{(ik_0)^2 - \vec{k}^2 - M^2 + i0^+} \\ &= \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 + M^2 - i0^+} \end{aligned}$$

$l^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$  denotes a Euclidian scalar product

- $I$  diverges for large values of  $l$  (ultraviolet divergence)
- $M^2 \rightarrow 0$ :  $I$  diverges for small values of  $l$  (infrared divergence)

The degree of divergence can be estimated by simply counting the powers of momenta.

If the integral behaves asymptotically as

$$\int d^4l/l^2 : \quad \text{diverges quadratically}$$
$$\int d^4l/l^3 : \quad \text{diverges linearly}$$
$$\int d^4l/l^4 : \quad \text{diverges logarithmically}$$

*I* diverges quadratically



**Dimensional regularization: Generalize from 4 to  $n$  dimensions and introduce polar coordinates**

$$\begin{aligned}l_1 &= l \cos(\theta_1) \\l_2 &= l \sin(\theta_1) \cos(\theta_2) \\l_3 &= l \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \\&\vdots \\l_{n-1} &= l \sin(\theta_1) \sin(\theta_2) \cdots \cos(\theta_{n-1}) \\l_n &= l \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-1})\end{aligned}$$

$$0 \leq l, \quad \theta_i \in [0, \pi], i = 1, \dots, n-2, \quad \theta_{n-1} \in [0, 2\pi]$$

**A general integral is then symbolically of the form**

$$\begin{aligned}\int d^n l \cdots &= \int_0^\infty l^{n-1} dl \\&\times \int_0^{2\pi} d\theta_{n-1} \int_0^\pi d\theta_{n-2} \sin(\theta_{n-2}) \cdots \int_0^\pi d\theta_1 \sin^{n-2}(\theta_1) \cdots\end{aligned}$$

If the integrand does not depend on the angles, the angular integration can explicitly be carried out:

$$\int d\Omega_n = 2 \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

**Example**

$$n = 3 : \quad 4\pi = 2 \frac{\pi}{1/2} = 2 \frac{\pi^{3/2}}{\sqrt{\pi}/2} = 2 \frac{\pi^{3/2}}{\Gamma(3/2)}$$

We define the integral for  $n$  dimensions ( $n$  integer) as

$$I_n(M^2, \mu^2) = \mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i0^+}$$

**Scale  $\mu$ : Unit of mass, 't Hooft parameter, renormalization scale (integral has the same dimension for arbitrary  $n$ )**

Integral formally reads

$$\begin{aligned}
 I_n(M^2, \mu^2) &= \mu^{4-n} \underbrace{2^{\frac{n}{2}} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}}_{\text{angular integration}} \frac{1}{(2\pi)^n} \underbrace{\int_0^\infty dl \frac{l^{n-1}}{l^2 + M^2}}_{\text{elementary}} \\
 &= \mu^{4-n} 2^{\frac{n}{2}} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{(2\pi)^n} \frac{1}{2} (M^2)^{\frac{n}{2}-1} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(1 - \frac{n}{2}\right)}{\underbrace{\Gamma(1)}_1} \\
 &= \frac{\mu^{4-n}}{(4\pi)^{\frac{n}{2}}} (M^2)^{\frac{n}{2}-1} \Gamma\left(1 - \frac{n}{2}\right)
 \end{aligned}$$

$\Gamma(z)$  is single valued and analytic over the entire complex plane, save for the points  $z = -n$ ,  $n = 0, 1, 2, \dots$ , where it possesses simple poles with residue  $(-1)^n/n!$

$a^z = \exp[\ln(a)z]$ ,  $a \in R^+$  is an analytic function in  $C$

Define (as a function of a complex variable  $n$ )

$$I(M^2, \mu^2, n) = \frac{M^2}{(4\pi)^2} \left( \frac{4\pi\mu^2}{M^2} \right)^{2-\frac{n}{2}} \Gamma\left(1 - \frac{n}{2}\right)$$

As  $n \rightarrow 4$  Gamma function has a pole  $\Rightarrow I(M^2, \mu^2, n)$  has a pole

How is this pole is approached?

Important property:  $\Gamma(z + 1) = z\Gamma(z)$

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{\Gamma\left(1 - \frac{n}{2} + 1\right)}{1 - \frac{n}{2}} = \frac{\Gamma\left(2 - \frac{n}{2} + 1\right)}{\left(1 - \frac{n}{2}\right)\left(2 - \frac{n}{2}\right)} = \frac{\Gamma\left(1 + \frac{\epsilon}{2}\right)}{(-1)\left(1 - \frac{\epsilon}{2}\right)\frac{\epsilon}{2}}$$

where  $\epsilon \equiv 4 - n$ .

$$a^x = \exp[\ln(a)x] = 1 + \ln(a)x + O(x^2)$$

$$I(M^2, \mu^2, n) = \frac{M^2}{16\pi^2} \left[ -\frac{2}{\epsilon} \underbrace{-\Gamma'(1)}_{\gamma_E} - 1 - \ln(4\pi) + \ln\left(\frac{M^2}{\mu^2}\right) + O(\epsilon) \right]$$

## Summary

$$I(M^2, \mu^2, n) = \frac{M^2}{16\pi^2} \left[ R + \ln \left( \frac{M^2}{\mu^2} \right) \right] + O(n - 4)$$

where

$$\underbrace{\underbrace{R}_{\overline{\text{MS}}}}_{\overline{\text{MS}}} = \underbrace{\frac{2}{n-4}}_{\overline{\text{MS}}} - [\ln(4\pi) + \Gamma'(1)] - 1$$

## The Chiral Lagrangian at $\mathcal{O}(q^4)$

S. Weinberg, The Quantum Theory of Fields, Vol. I, Chap. 12

... the cancellation of ultraviolet divergences does not really depend on renormalizability; as long as we include every one of the infinite number of interactions allowed by symmetries, the so-called non-renormalizable theories are actually just as renormalizable as renormalizable theories.

**Conclusion: Adjust (renormalize) parameters of  $\mathcal{L}_4$  to cancel one-loop infinities!**

$$L_i = L_i^r + \frac{\Gamma_i}{32\pi^2} R, \quad i = 1, \dots, 10$$
$$H_i = H_i^r + \frac{\Delta_i}{32\pi^2} R, \quad i = 1, 2$$

$\mathcal{L}_4$  of Gasser and Leutwyler:<sup>17</sup>

$$\begin{aligned}
\mathcal{L}_4 = & L_1 \left\{ \text{Tr} [D_\mu U (D^\mu U)^\dagger] \right\}^2 + L_2 \text{Tr} [D_\mu U (D_\nu U)^\dagger] \text{Tr} [D^\mu U (D^\nu U)^\dagger] \\
& + L_3 \text{Tr} [D_\mu U (D^\mu U)^\dagger D_\nu U (D^\nu U)^\dagger] \\
& + L_4 \text{Tr} [D_\mu U (D^\mu U)^\dagger] \text{Tr} (\chi U^\dagger + U \chi^\dagger) \\
& + L_5 \text{Tr} [D_\mu U (D^\mu U)^\dagger (\chi U^\dagger + U \chi^\dagger)] \\
& + L_6 \left[ \text{Tr} (\chi U^\dagger + U \chi^\dagger) \right]^2 + L_7 \left[ \text{Tr} (\chi U^\dagger - U \chi^\dagger) \right]^2 \\
& + L_8 \text{Tr} (U \chi^\dagger U \chi^\dagger + \chi U^\dagger \chi U^\dagger) \\
& - i L_9 \text{Tr} \left[ f_{\mu\nu}^R D^\mu U (D^\nu U)^\dagger + f_{\mu\nu}^L (D^\mu U)^\dagger D^\nu U \right] \\
& + L_{10} \text{Tr} (U f_{\mu\nu}^L U^\dagger f_R^{\mu\nu}) \\
& + H_1 \text{Tr} (f_{\mu\nu}^R f_R^{\mu\nu} + f_{\mu\nu}^L f_L^{\mu\nu}) \\
& + H_2 \text{Tr} (\chi \chi^\dagger)
\end{aligned}$$

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<sup>17</sup>J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985)

Coefficient	Empirical Value	$\Gamma_i$
$L_1^r$	$0.4 \pm 0.3$	$\frac{3}{32}$
$L_2^r$	$1.35 \pm 0.3$	$\frac{3}{16}$
$L_3^r$	$-3.5 \pm 1.1$	0
$L_4^r$	$-0.3 \pm 0.5$	$\frac{1}{8}$
$L_5^r$	$1.4 \pm 0.5$	$\frac{3}{8}$
$L_6^r$	$-0.2 \pm 0.3$	$\frac{11}{144}$
$L_7^r$	$-0.4 \pm 0.2$	0
$L_8^r$	$0.9 \pm 0.3$	$\frac{5}{48}$
$L_9^r$	$6.9 \pm 0.7$	$\frac{1}{4}$
$L_{10}^r$	$-5.5 \pm 0.7$	$-\frac{1}{4}$

The renormalized coefficients  $L_i^r$  depend on the scale  $\mu$  introduced by dimensional regularization and their values at two different scales  $\mu_1$  and  $\mu_2$  are related by

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln \left( \frac{\mu_1}{\mu_2} \right)$$



## Present status of the mesonic Lagrangian $[\text{SU}(3) \times \text{SU}(3)]^{18} (\pi, K, \eta)$

$$\underbrace{2}_{\mathcal{O}(q^2)} + \underbrace{10 + 2}_{\mathcal{O}(q^4)} + \underbrace{90 + 4 + 23}_{\mathcal{O}(q^6)} + \dots$$

- $q$ : Small quantity such as a pion mass
- Even powers
- Two-loop level

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<sup>18</sup>J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985);  
H. W. Fearing and S. Scherer, Phys. Rev. D 53, 315 (1996);  
J. Bijnens, G. Colangelo, G. Ecker, JHEP 02, 020 (1999);  
T. Ebertshäuser, H. W. Fearing, S. Scherer, Phys. Rev. D 65, 054033 (2002);  
J. Bijnens, L. Girlanda, P. Talavera, Eur. Phys. J. C 23, 539 (2002)

## Masses at $\mathcal{O}(q^4)$

Definition of the propagator of a (pseudo-) scalar field:

$$i\Delta(p) = \int d^4x e^{-ip \cdot x} \langle 0 | T [\Phi_0(x) \Phi_0(0)] | 0 \rangle$$

index 0: bare unrenormalized field

At lowest order ( $D = 2$ ) the propagator simply reads

$$i\Delta(p) = \frac{i}{p^2 - M_0^2 + i0^+}$$

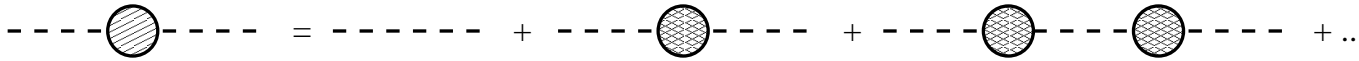
with lowest-order masses  $M_0$

$$M_{\pi,2}^2 = 2B_0 \hat{m}$$

$$M_{K,2}^2 = B_0 (\hat{m} + m_s)$$

$$M_{\eta,2}^2 = \frac{2}{3} B_0 (\hat{m} + 2m_s)$$

Full propagator in terms of the so-called proper self-energy insertions  $-i\Sigma(p^2)$



## Summation via a geometric series

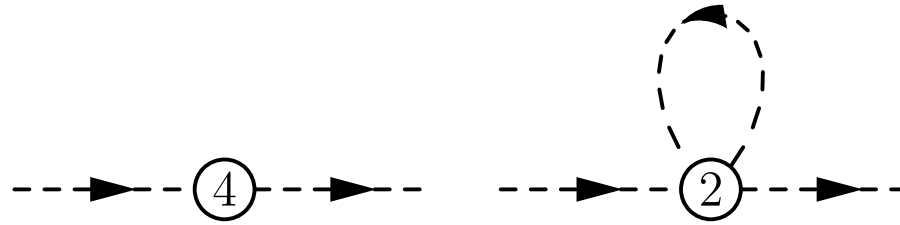
$$\begin{aligned}
 i\Delta(p) &= \frac{i}{p^2 - M_0^2 + i0^+} \\
 &+ \frac{i}{p^2 - M_0^2 + i0^+} \underbrace{[-i\Sigma(p^2)]}_{x} \frac{i}{p^2 - M_0^2 + i0^+} \\
 &+ \dots \\
 &= \frac{i}{p^2 - M_0^2 + i0^+} \underbrace{[1 + x + x^2 + \dots]}_{1/(1-x)} \\
 &= \frac{i}{p^2 - M_0^2 - \Sigma(p^2) + i0^+}
 \end{aligned}$$

$-i\Sigma(p^2)$ : one-particle-irreducible diagrams

## Definition of the physical mass

$$M^2 - M_0^2 - \Sigma(M^2) \stackrel{!}{=} 0$$

Self-energy contributions at  $D = 4$ :



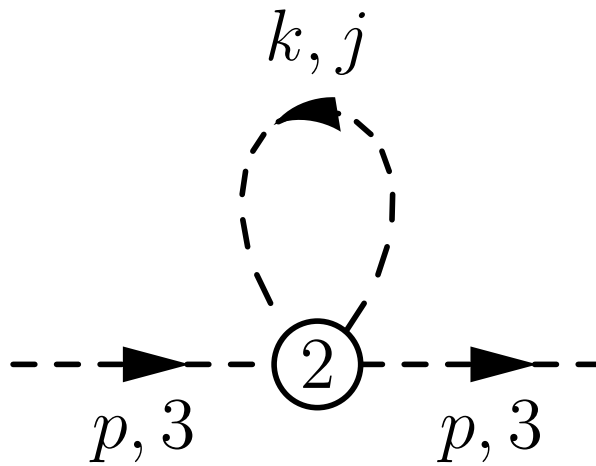
$$\mathcal{L}_{\text{int}} = \mathcal{L}_4^{2\phi} + \mathcal{L}_2^{4\phi}$$

At  $\mathcal{O}(q^4)$  the self-energies are of the form

$$\Sigma_\phi(p^2) = A_\phi + B_\phi p^2$$

$A_\phi$  and  $B_\phi$  receive a tree-level contribution from  $\mathcal{L}_4$  and a one-loop contribution with a vertex from  $\mathcal{L}_2$

Example: pion-loop contribution to the  $\pi^0$  self-energy



$$\frac{i}{6F_0^2}(-4p^2 + M_{\pi,2}^2)I(M_{\pi,2}^2, \mu^2, n)$$

diverges as  $n \rightarrow 4$

Example:

$$A_\pi = \frac{M_{\pi,2}^2}{F_0^2} \left\{ \underbrace{-\frac{1}{6}I(M_{\pi,2}^2) - \frac{1}{6}I(M_{\eta,2}^2) - \frac{1}{3}I(M_{K,2}^2)}_{\text{one-loop contribution}} \right. \\ \left. \underbrace{+32[(2m + m_s)B_0L_6 + mB_0L_8]}_{\text{contact contribution}} \right\}$$

$$B_\pi = \frac{2I(M_{\pi,2}^2)}{3F_0^2} + \frac{1I(M_{K,2}^2)}{3F_0^2} - \frac{16B_0}{F_0^2} [(2m + m_s)L_4 + mL_5]$$

Masses at  $\mathcal{O}(q^4)$

$$\begin{aligned} M^2 &= M_0^2 + \Sigma(M^2) = M_0^2 + A + BM^2 \\ &= \frac{M_0^2 + A}{1 - B} = M_0^2(1 + B) + A + \mathcal{O}(q^6) \end{aligned}$$

because  $A = \mathcal{O}(q^4)$  and  $\{B, M_0^2\} = \mathcal{O}(q^2)$

## Final result

$$M_{\pi,4}^2 = M_{\pi,2}^2 \left\{ 1 + \frac{M_{\pi,2}^2}{32\pi^2 F_0^2} \ln \left( \frac{M_{\pi,2}^2}{\mu^2} \right) - \frac{M_{\eta,2}^2}{96\pi^2 F_0^2} \ln \left( \frac{M_{\eta,2}^2}{\mu^2} \right) + \frac{16}{F_0^2} [(2m + m_s) B_0 (2L_6^r - L_4^r) + m B_0 (2L_8^r - L_5^r)] \right\}$$

## Remarks:

1. Expressions for the masses are finite. The bare coefficients  $L_i$  of the Lagrangian of Gasser and Leutwyler must be infinite in order to cancel the infinities resulting from the divergent loop integrals.
2. At any order  $\mathcal{O}(q^{2n})$  the masses of the Goldstone bosons vanish, if the quark masses are sent to zero.

3. The quark masses appear in combination with  $B_0$ . No absolute statements about quark masses possible without knowledge about  $B_0$ .
4. Analytic terms  $\sim m_q^2$  multiplied by the renormalized low-energy coupling constants  $L_i^r$ .
5. Non-analytic terms  $\sim m_q^2 \ln(m_q)$  (so-called chiral logarithms) contain no new constants.
6. Physical observables do not depend on the scale  $\mu$ .



## Chiral Perturbation Theory for Baryons

- Interaction of pions and nucleons<sup>19</sup>
- Most general Lagrangian

$$\mathcal{L}_{\pi N} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \dots$$

## Transformation properties of the fields

### Nucleon field

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

### Transformation behavior under isospin SU(2)

$$\Psi \mapsto V\Psi$$

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<sup>19</sup>J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988);  
A. Krause, Helv. Phys. Acta 63, 3 (1990)

## Introduce

$$u^2(x) = U(x)$$

Define  $K(V_L, V_R, U)$  in terms of

$$u(x) \mapsto u'(x) = \sqrt{V_R U V_L^\dagger} \equiv V_R u K^{-1}(V_L, V_R, U)$$

i.e.

$$K(V_L, V_R, U) = u'^{-1} V_R u = \sqrt{V_R U V_L^\dagger}^{-1} V_R \sqrt{U}$$

Transformation properties under local  $\mathbf{SU}(2)_L \times \mathbf{SU}(2)_R \times \mathbf{U}(1)_V$

$$\begin{pmatrix} U \\ \Psi \end{pmatrix} \mapsto \begin{pmatrix} U' \\ \Psi' \end{pmatrix} = \begin{pmatrix} V_R U V_L^\dagger \\ \exp[-i\Theta] K(V_L, V_R, U) \Psi \end{pmatrix}$$

**Exercise:** Verify

$$K(V'_L, V'_R, V_R U V_L^\dagger) K(V_L, V_R, U) = K((V'_L V_L), (V'_R V_R), U)$$

We need a covariant derivative of the nucleon field with

$$D'_\mu \Psi' = \exp(-i\Theta) K(V_L, V_R, U) D_\mu \Psi \quad (*)$$

Introduce

$$\Gamma_\mu = \frac{1}{2} \left[ u^\dagger (\partial_\mu - ir_\mu) u + u (\partial_\mu - il_\mu) u^\dagger \right]$$

and define

$$D_\mu \Psi = (\partial_\mu + \Gamma_\mu - iv_\mu^{(s)}) \Psi$$

**Exercise:** Verify (\*)

Define

$$u_\mu \equiv i \left[ u^\dagger (\partial_\mu - ir_\mu) u - u (\partial_\mu - il_\mu) u^\dagger \right]$$

Behavior under parity

$$u_\mu \xrightarrow{P} i \left[ u (\partial^\mu - il^\mu) u^\dagger - u^\dagger (\partial^\mu - ir^\mu) u \right] = -u^\mu$$

**Exercise:** Using

$$u' = V_R u K^\dagger = K u V_L^\dagger$$

show that, under  $SU(2)_L \times SU(2)_R \times U(1)_V$ ,  $u_\mu$  transforms as

$$u_\mu \mapsto K u_\mu K^\dagger$$

Counting scheme for the (new) elements of baryon chiral perturbation theory<sup>20</sup>

$$\Psi, \bar{\Psi} = \mathcal{O}(q^0)$$

$$D_\mu \Psi = \mathcal{O}(q^0)$$

$$(i\not{D} - m)\Psi = \mathcal{O}(q)$$

$$1, \gamma_\mu, \gamma_5 \gamma_\mu, \sigma_{\mu\nu} = \mathcal{O}(q^0)$$

$$\gamma_5 = \mathcal{O}(q)$$

order given is the minimal one

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<sup>20</sup>A. Krause, *Helv. Phys. Acta* 63, 3 (1990)

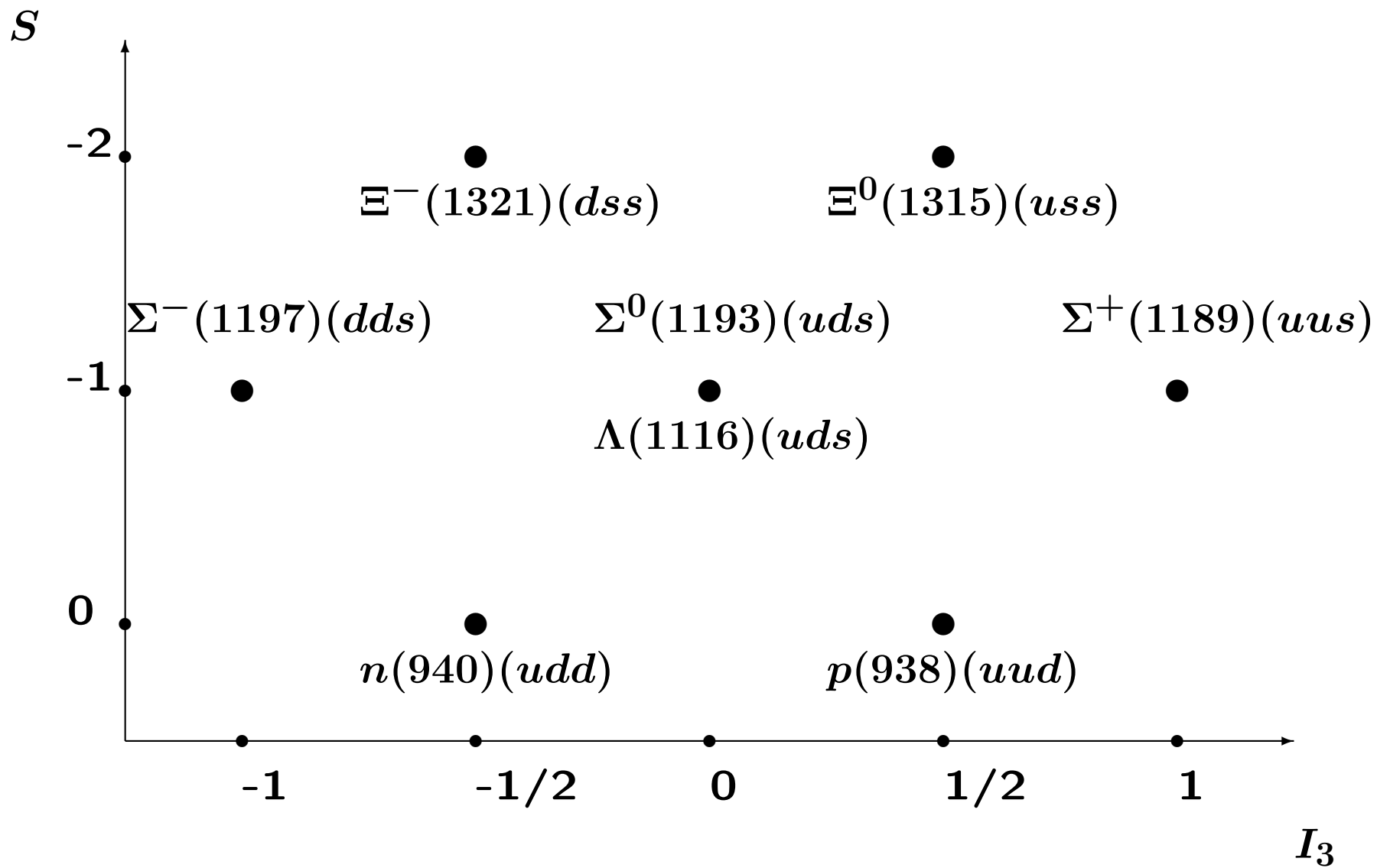
$$\begin{aligned}
\mathcal{L}_{\pi N}^{(1)} &= \bar{\Psi} \left( i\not{D} - m + \frac{\overset{\circ}{g}_A}{2} \gamma^\mu \gamma_5 u_\mu \right) \Psi \\
&= \bar{\Psi} \left( i\gamma_\mu \partial^\mu - \boxed{m} - \frac{1}{2} \frac{\boxed{g_A}}{F} \gamma_\mu \gamma_5 \tau^a \partial^\mu \pi^a \right) \Psi + \dots
\end{aligned}$$

Two parameters not determined by chiral symmetry:

- nucleon mass  $m$  in the chiral limit
- axial-vector coupling constant  $g_A$  in the chiral limit

[Physical nucleon mass:  $m_N = 939$  MeV. Theoretical analysis:  $m \approx 883$  MeV (at fixed  $m_s \neq 0$ ). Physical axial-vector coupling constant from neutron beta decay:  $g_A = 1.267$ .]

# SU(3)



$$B = \sum_{a=1}^8 \frac{\lambda_a B_a}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & & \Sigma^+ & & p \\ & \Sigma^- & & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ & \Xi^- & & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}$$

Transformation behavior under flavor  $\mathbf{SU}(3)_V$

$$B \mapsto V B V^\dagger$$

Transformation behavior under  $\mathbf{SU}(3)_L \times \mathbf{SU}(3)_R$

$$\begin{pmatrix} U \\ B \end{pmatrix} \mapsto \begin{pmatrix} U' \\ B' \end{pmatrix} = \begin{pmatrix} V_R U V_L^\dagger \\ K(V_L, V_R, U) B K^\dagger(V_L, V_R, U) \end{pmatrix}$$

Covariant derivative

$$D_\mu B = \partial_\mu B + [\Gamma_\mu, B]$$

Most general Lagrangian at  $\mathcal{O}(q)$

$$\mathcal{L}_{MB}^{(1)} = \text{Tr} [\bar{B} (i\not{D} - M_0) B] - \frac{D}{2} \text{Tr} (\bar{B} \gamma^\mu \gamma_5 \{u_\mu, B\}) - \frac{F}{2} \text{Tr} (\bar{B} \gamma^\mu \gamma_5 [u_\mu, B])$$

Three parameters not determined by chiral symmetry:

- Mass of the baryon octet in the chiral limit  $M_0$
- $D$  and  $F$  may be determined by fitting the semi-leptonic decays  $B \rightarrow B' + e^- + \bar{\nu}_e$  at tree level:

$$D = 0.80, \quad F = 0.50$$



## Present status of the baryonic Lagrangian $[\text{SU}(2) \times \text{SU}(2) \times \text{U}(1)]^{21}$ ( $\pi, N$ )

$$\underbrace{2}_{\mathcal{O}(q)} + \underbrace{7}_{\mathcal{O}(q^2)} + \underbrace{23}_{\mathcal{O}(q^3)} + \underbrace{118}_{\mathcal{O}(q^4)} + \dots$$

- Odd and even powers (spin)
- One-loop level

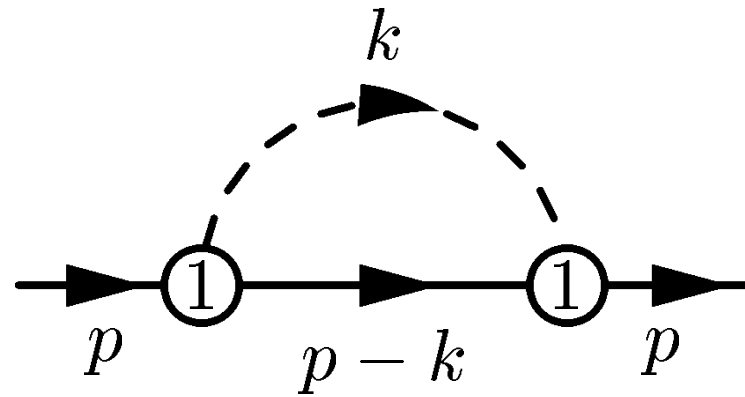
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<sup>21</sup>J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988);  
V. Bernard, N. Kaiser, U.-G. Meißner, Int. J. Mod. Phys. E 4, 193 (1995);  
G. Ecker and M. Mojžiš, Phys. Lett. B 365, 312 (1996);  
N. Fettes, U.-G. Meißner, M. Mojžiš, S. Steininger, Ann. Phys. (N.Y.) 283, 273 (2000)

## Power counting and renormalization: Outline of the problem

- **Power counting:** Associate chiral order  $D$  with a diagram
  - Loop integration in  $n$  dimensions  $\sim \mathcal{O}(q^n)$
  - Vertex from  $\mathcal{L}_{2k} \sim \mathcal{O}(q^{2k})$
  - Vertex from  $\mathcal{L}_{\pi N}^{(k)} \sim \mathcal{O}(q^k)$
  - Nucleon propagator  $\sim \mathcal{O}(q^{-1})$
  - Pion propagator  $\sim \mathcal{O}(q^{-2})$

- Example: Contribution to nucleon mass



**Goal:**  $D = n \cdot 1 - 2 \cdot 1 - 1 \cdot 1 + 2 \cdot 1 = n - 1$

**Result**

$$\Sigma = -\frac{3g_A^2}{4F^2} [(\not{p} + m)I_N + M^2(\not{p} + m)I_{N\pi}(-p, 0) + \dots]$$

Apply  $\widetilde{\text{MS}}$  renormalization scheme

$$\Sigma_r = -\frac{3g_{Ar}^2}{4F_r^2} [M^2(\not{p} + m) \underbrace{I_{N\pi}^r(-p, 0)}_{-\frac{1}{16\pi^2} + \dots} + \dots] = \mathcal{O}(q^2)$$

**GSS:**<sup>22</sup> It turns out that loops have a much more complicated low-energy structure if baryons are included. Because the nucleon mass  $m_N$  does not vanish in the chiral limit, the mass scale  $m$  (nucleon mass in the chiral limit) occurs in the effective Lagrangian  $\mathcal{L}_{\pi N}^{(1)} \dots$

**This complicates life a lot.**

**BKKM:**<sup>23</sup> Stated differently, the consistent power counting scheme present in the mesonic sector is altered when baryons are included and one no longer has a one-to-one mapping between the loop and small-momentum expansion.

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<sup>22</sup>J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988)

<sup>23</sup>V. Bernard, N. Kaiser, J. Kambor, U.-G. Meißner, Nucl. Phys. B388, 315 (1992)

## Solutions

### Solution 1: Heavy-Baryon Approach <sup>24</sup>

- Trick: Let  $p = mv + k_p$ ,  $v^2 = 1$ ,  $v^0 \geq 1$  [Often  $v^\mu = (1, 0, 0, 0)$ ]

$$\Psi(x) = e^{-imv \cdot x} (\mathcal{N}_v + \mathcal{H}_v)$$

with

$$\begin{aligned}\mathcal{N}_v &= e^{+imv \cdot x} \frac{1}{2} (1 + \not{v}) \Psi \\ \mathcal{H}_v &= e^{+imv \cdot x} \frac{1}{2} (1 - \not{v}) \Psi\end{aligned}$$

- Using the equation of motion for  $\mathcal{H}_v$ , one can eliminate  $\mathcal{H}_v$  and obtain a Lagrangian for  $\mathcal{N}_v$

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<sup>24</sup>E. Jenkins and A. V. Manohar, Phys. Lett. B 255, 558 (1991)

- To lowest order

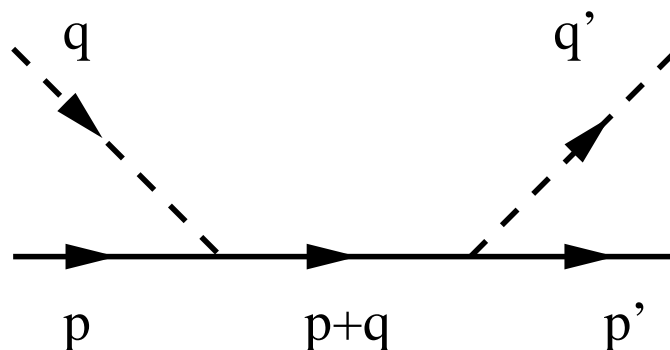
$$\widehat{\mathcal{L}}_{\pi N}^{(1)} = \bar{\mathcal{N}}_v (i v \cdot D + g_A S_v \cdot u) \mathcal{N}_v + \mathcal{O}(1/m)$$

**$1/m$  expansion of the Lagrangian similar to a Foldy-Wouthuysen expansion**

- Power counting works as in the mesonic sector ( $\widetilde{\text{MS}}$  scheme)
- But ...

- ... problems with analyticity

Simple example (T. Becher, Chiral Dynamics 2000)



Singularity due to the nucleon pole in the  $s$  channel is understood in terms of the relativistic propagator

$$\frac{1}{(p+q)^2 - m_N^2} = \frac{1}{2p \cdot q + M_\pi^2}$$

pole at  $2p \cdot q = -M_\pi^2$

Heavy-baryon type of expansion (with  $p^\mu = m_N v^\mu$ )

$$\begin{aligned} \frac{1}{2p \cdot q + M_\pi^2} &= \frac{1}{2m_N} \frac{1}{v \cdot q + \frac{M_\pi^2}{2m_N}} \\ &= \frac{1}{2m_N} \frac{1}{v \cdot q} \left( 1 - \frac{M_\pi^2}{2m_N v \cdot q} + \dots \right) \end{aligned}$$

To any finite order the heavy-baryon expansion produces poles at

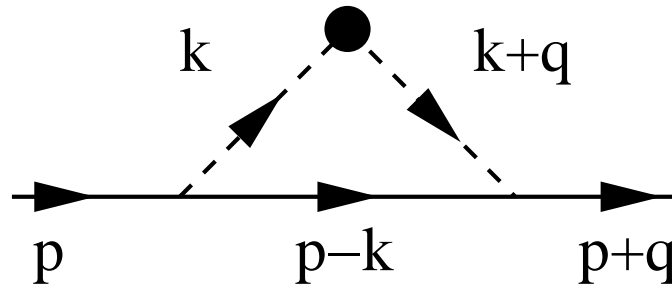
$$v \cdot q = 0$$

instead of a simple pole at

$$v \cdot q = -M_\pi^2/(2m_N)$$



## Second example: Triangle diagram



- Problems in the analytic behavior of form factors in the time-like region
- Example: Scalar form factor

## Solution 2: Infrared regularization<sup>25</sup>

Special treatment of one-loop integrals based on the Feynman parametrization

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}$$

$$a = (p - k)^2 - m^2 + i0^+$$

$$b = k^2 - M^2 + i0^+$$

$$H = \int_0^1 dz \cdots = \int_0^\infty dz \cdots - \int_1^\infty dz \cdots \equiv I + R$$

- $I$ : power counting o.k.
- $R$ : violates power counting; regular, i.e., can be absorbed in counterterms

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<sup>25</sup>T. Becher and H. Leutwyler, Eur. Phys. J. C 9, 643 (1999)

### Solution 3: **Extended on-mass-shell (EOMS) scheme**<sup>26</sup>

Main idea: Perform **additional subtractions** such that **renormalized diagrams** satisfy the power counting

Motivation for this approach<sup>27</sup>

Terms violating the power counting are **analytic** in small quantities (and can thus be absorbed in a renormalization of counterterms)

- Example (chiral limit)

$$H(p^2, m^2; n) = - \int \frac{d^n k}{(2\pi)^n} \frac{i}{[(k-p)^2 - m^2 + i0^+][k^2 + i0^+]}$$

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<sup>26</sup>T. Fuchs, J. Gegelia, G. Japaridze, S. Scherer, Phys. Rev. D 68, 056005 (2003)

<sup>27</sup>J. Gegelia and G. Japaridze, Phys. Rev. D 60, 114038 (1999)

Small quantity

$$\Delta = \frac{p^2 - m^2}{m^2} = \mathcal{O}(q)$$

We want the **renormalized** integral to be of order

$$D = n - 1 - 2 = n - 3$$

Result of integration

$$H \sim F(n, \Delta) + \Delta^{n-3} G(n, \Delta)$$

$F$  and  $G$  are hypergeometric functions; **analytic** in  $\Delta$  for arbitrary  $n$

Observation<sup>28</sup>

$F$  corresponds to **first** expanding the integrand in small quantities and **then** performing the integration

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<sup>28</sup>J. Gegelia, G. Japaridze, K. S. Turashvili, Theor. Math. Phys. 101, 1313 (1994)

⇒ **Algorithm**: Expand integrand in small quantities and subtract those (integrated) terms whose order is **smaller** than suggested by the power counting

Here:

$$\begin{aligned}
 H^{\text{subtr}} &= - \int \frac{d^n k}{(2\pi)^n} \frac{i}{(k^2 - 2k \cdot p + i0^+)(k^2 + i0^+)} \Big|_{p^2=m^2} \\
 &= -2\bar{\lambda} + \frac{1}{16\pi^2} + O(n-4)
 \end{aligned}$$

where

$$\bar{\lambda} = \frac{m^{n-4}}{(4\pi)^2} \left\{ \frac{1}{n-4} - \frac{1}{2} [\ln(4\pi) + \Gamma'(1) + 1] \right\}$$

$$H^R = H - H^{\text{subtr}} = \mathcal{O}(q^{n-3})$$

## General case including pion mass

$$\begin{aligned} & \left. \frac{1}{(k^2 - 2k \cdot p + i0^+) (k^2 + i0^+)} \right|_{p^2=m^2} \\ & + (p^2 - m^2) \left[ \frac{1}{2m^2} \frac{1}{(k^2 - 2k \cdot p + i0^+)^2} + \dots \right]_{p^2=m^2} \\ & + M^2 \left. \frac{1}{(k^2 - 2k \cdot p + i0^+) (k^2 + i0^+)^2} \right|_{p^2=m^2} \\ & + \dots \end{aligned}$$

## Remarks:

- **(Axial) Vector mesons can be consistently included<sup>29</sup>**
  - Improved phenomenology<sup>30</sup>
  - Larger radius of convergence
  
- **IR renormalization can be reformulated in terms of EOMS<sup>31</sup>**
  - Formal equivalence shown at one-loop level
  - Known integrals tested

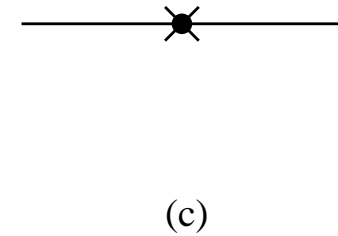
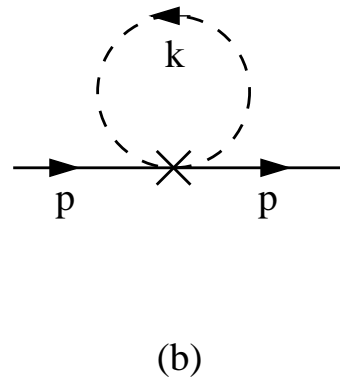
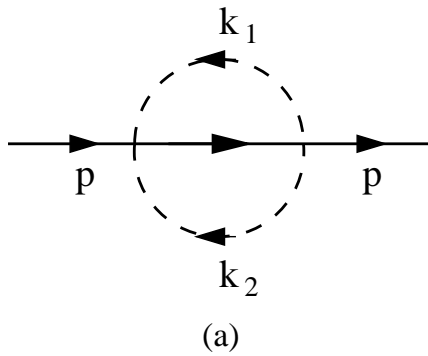
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<sup>29</sup>T. Fuchs, M. R. Schindler, J. Gegelia, S. Scherer, Phys. Lett. B 575, 11 (2003)

<sup>30</sup>M. R. Schindler, J. Gegelia, S. Scherer, Eur. Phys. J. A 26, 1 (2005)

<sup>31</sup>M. R. Schindler, J. Gegelia, S. Scherer, Phys. Lett. B 586, 258 (2004)

- One two-loop example has been explicitly tested<sup>32</sup>



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<sup>32</sup>M. R. Schindler, J. Gegelia, S. Scherer, Nucl. Phys. B682, 367 (2004)



## The EOMS approach in more detail

Calculation of the nucleon mass up to and including order  $\mathcal{O}(q^3)$

Full propagator

$$S_0(p) = \frac{1}{\not{p} - m_0 - \Sigma_0(\not{p})} \equiv \frac{1}{\not{p} - m - \Sigma(\not{p})}$$

- $m_0$  bare mass
- $m$  nucleon mass in the chiral limit
- $\Sigma_0(\not{p})$  self energy

Definition of the nucleon mass

$$m_N - m_0 - \Sigma_0(m_N) = m_N - m - \Sigma(m_N) = 0$$

- Tree-level contribution

Recall  $\pi N$  Lagrangian at order  $\mathcal{O}(q^2)$

$$\begin{aligned} \mathcal{L}_{\pi N}^{(2)} = & c_1 \text{Tr}(\chi_+) \bar{\Psi} \Psi - \frac{c_2}{4m^2} [\bar{\Psi} \text{Tr}(u_\mu u_\nu) D^\mu D^\nu \Psi + \text{H.c.}] \\ & + \bar{\Psi} \left[ \frac{c_3}{2} \text{Tr}(u_\mu u^\mu) + i \frac{c_4}{4} [u_\mu, u_\nu] + c_5 \left[ \chi_+ - \frac{1}{2} \text{Tr}(\chi_+) \right] \right. \\ & \left. + \frac{c_6}{2} f_{\mu\nu}^+ + \frac{c_7}{2} v_{\mu\nu}^{(s)} \right] \sigma^{\mu\nu} \Psi \end{aligned}$$

Only  $c_1$  term contributes to the self energy

$$-4c_1 M^2$$

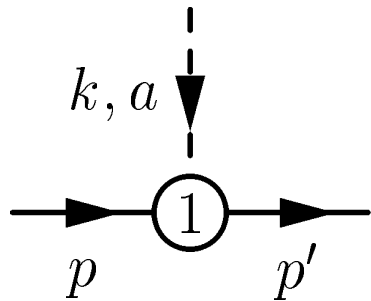
No contact contributions from the Lagrangian  $\mathcal{L}_{\pi N}^{(3)}$

- Loop contributions

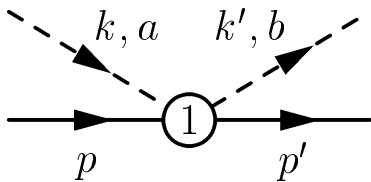
Expand  $\mathcal{L}_{\pi N}^{(1)}$  up to and including two pion fields:

$$\mathcal{L}_{\text{int}}^{(1)} = -\frac{1}{2} \frac{g_A}{F} \bar{\Psi} \gamma^\mu \gamma_5 \tau^b \partial_\mu \phi^b \Psi - \frac{1}{4F^2} \bar{\Psi} \gamma^\mu \vec{\tau} \cdot \vec{\phi} \times \partial_\mu \vec{\phi} \Psi$$

Feynman rules

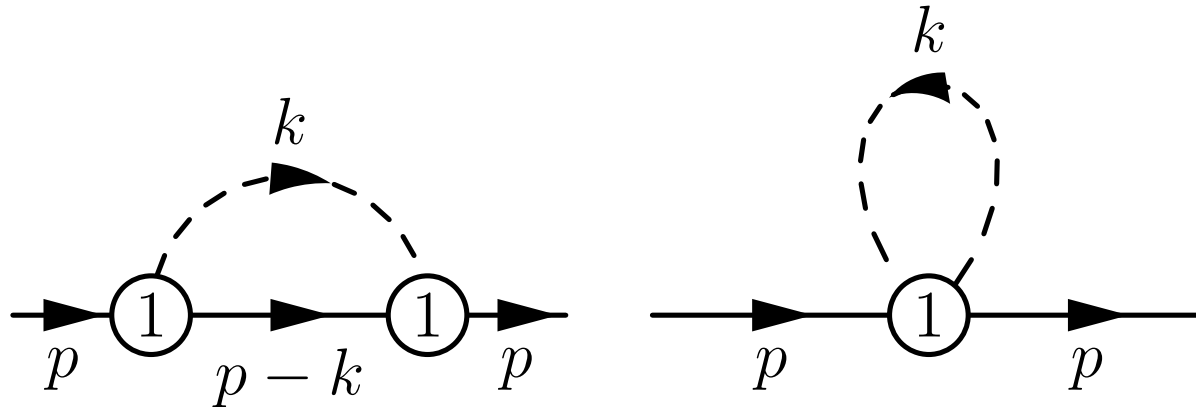


$$-\frac{g_A}{2F} \not{k} \gamma_5 \tau_a$$



$$\frac{1}{4F^2} (\not{k} + \not{k}') \epsilon_{abc} \tau_c$$

Two types of loop contributions at order  $\mathcal{O}(q^3)$



Second diagram does not contribute:  $\epsilon_{aac} = 0$

Feynman rules + propagators +  $\tau_a \tau_a = 3$

$$i\Delta_\pi(p) = \frac{i}{p^2 - M^2 + i0^+}$$

$$iS_N(p) = i \frac{\not{p} + m - i0^+}{p^2 - m^2 + i0^+}$$

⇒ contribution of the first diagram in dim. reg.

$$-i\Sigma^{\text{loop}}(\not{p}) = -i\frac{3g_A^2}{4F^2} i\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{\not{k}(\not{p} - m - \not{k})\not{k}}{[(p - k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]}$$

Simplify numerator using  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

$$-(\not{p} + m) \frac{\underbrace{k^2}}{k^2 - M^2 + M^2} + (p^2 - m^2)\not{k} - [(p - k)^2 - m^2]\not{k}$$

## Intermediate result

$$\begin{aligned}
 \Sigma^{\text{loop}}(\not{p}) = & \frac{3g_A^2}{4F^2} \left\{ -(\not{p} + m)\mu^{4-n_i} \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(p - k)^2 - m^2 + i0^+]} \right. \\
 & - (\not{p} + m)M^2\mu^{4-n_i} \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(p - k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} \\
 & + (p^2 - m^2)\mu^{4-n_i} \int \frac{d^n k}{(2\pi)^n} \frac{\not{k}}{[(p - k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} \\
 & \left. - \mu^{4-n_i} \int \frac{d^n k}{(2\pi)^n} \frac{\not{k}}{[k^2 - M^2 + i0^+]} \right\}
 \end{aligned}$$

Last term vanishes (integrand odd)

## Convention

$$\begin{aligned}
 I_{N\dots\pi\dots}(p_1, \dots, q_1, \dots) \\
 = \mu^{4-n_i} \int \frac{d^n k}{(2\pi)^n} \frac{1}{[(k + p_1)^2 - m^2 + i0^+] \dots [(k + q_1)^2 - M^2 + i0^+] \dots}
 \end{aligned}$$

To determine the vector integral use the ansatz

$$\mu^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{[(p-k)^2 - m^2 + i0^+][k^2 - M^2 + i0^+]} = p_\mu C$$

Multiply by  $p^\mu \Rightarrow$

$$C = \frac{1}{2p^2} \left[ I_N - I_\pi + (p^2 - m^2 + M^2) I_{N\pi}(-p, 0) \right]$$

Using the above convention the loop contribution to the nucleon self energy reads

$$\Sigma^{\text{loop}}(\not{p}) = -\frac{3g_A^2}{4F^2} \left\{ (\not{p} + m) I_N + (\not{p} + m) M^2 I_{N\pi}(-p, 0) - (p^2 - m^2) \frac{\not{p}}{2p^2} \left[ I_N - I_\pi + (p^2 - m^2 + M^2) I_{N\pi}(-p, 0) \right] \right\}$$

The explicit expressions for the integrals are given by

$$\begin{aligned}
 I_\pi &= \frac{M^2}{16\pi^2} \left[ R + \ln \left( \frac{M^2}{\mu^2} \right) \right] \\
 I_N &= \frac{m^2}{16\pi^2} \left[ R + \ln \left( \frac{m^2}{\mu^2} \right) \right] \\
 I_{N\pi}(p, 0) &= \frac{1}{16\pi^2} \left[ R + \ln \left( \frac{m^2}{\mu^2} \right) - 1 \right. \\
 &\quad \left. + \frac{p^2 - m^2 + M^2}{p^2} \ln \left( \frac{M}{m} \right) + \frac{2mM}{p^2} F(\Omega) \right]
 \end{aligned}$$

where

$$\begin{aligned}
 R &= \frac{2}{n-4} - [\ln(4\pi) + \Gamma'(1) + 1] \\
 \Omega &= \frac{p^2 - m^2 - M^2}{2mM}
 \end{aligned}$$



and

$$F(\Omega) = \begin{cases} \sqrt{\Omega^2 - 1} \ln \left( -\Omega - \sqrt{\Omega^2 - 1} \right), & \Omega \leq -1, \\ \sqrt{1 - \Omega^2} \arccos(-\Omega), & -1 \leq \Omega \leq 1, \\ \sqrt{\Omega^2 - 1} \ln \left( \Omega + \sqrt{\Omega^2 - 1} \right) - i\pi\sqrt{\Omega^2 - 1}, & 1 \leq \Omega. \end{cases}$$

- $\Sigma^{\text{loop}}$  contains divergences as  $n \rightarrow 4$  (the terms proportional to  $R$ )  $\Rightarrow$  needs to be renormalized
- For convenience:  $\mu = m$
- $\widetilde{\text{MS}}$  renormalization:
  - drop terms proportional to  $R$
  - replace all bare coupling constants ( $c_{1,0}, g_{A0}, F_0$ ) with the renormalized ones, now indicated by a subscript  $r$

⇒  $\widetilde{\text{MS}}$  renormalized self energy contribution

$$\Sigma_r^{\text{loop}}(\not{p}) = -\frac{3g_{Ar}^2}{4F_r^2} \left\{ (\not{p} + m)M^2 I_{N\pi}^r(-p, 0) - (p^2 - m^2) \frac{\not{p}}{2p^2} \left[ (p^2 - m^2 + M^2) I_{N\pi}^r(-p) - I_\pi^r \right] \right\}$$

Using

$$I_{N\pi}^r(-p, 0) = -\frac{1}{16\pi^2} + \dots$$

⇒ contribution of  $\mathcal{O}(q^2)$

Solve for the nucleon mass

$$\begin{aligned} m_N &= m + \Sigma_r^{\text{contact}}(m_N) + \Sigma_r^{\text{loop}}(m_N) \\ &= m - 4c_{1r}M^2 + \Sigma_r^{\text{loop}}(m_N) \end{aligned}$$

- $m_N - m = \mathcal{O}(q^2)$

- We need  $\Sigma_r^{\text{loop}}(m_N)$  to  $\mathcal{O}(q^3)$

- Expansion of  $I_{N\pi}^r$

$$\arccos(-\Omega) = \frac{\pi}{2} + \dots$$

$$I_{N\pi}^r = \frac{1}{16\pi^2} \left( -1 + \frac{\pi M}{m} + \dots \right)$$

- This yields

$$m_N = m - 4c_{1r}M^2 + \boxed{\frac{3g_{Ar}^2 M^2}{32\pi^2 F_r^2} m} - \frac{3g_{Ar}^2 M^3}{32\pi F_r^2}$$

- Power counting problem

## Solution

Term violating the power counting is analytic in small quantities and can thus be absorbed in counter terms

## Rewrite

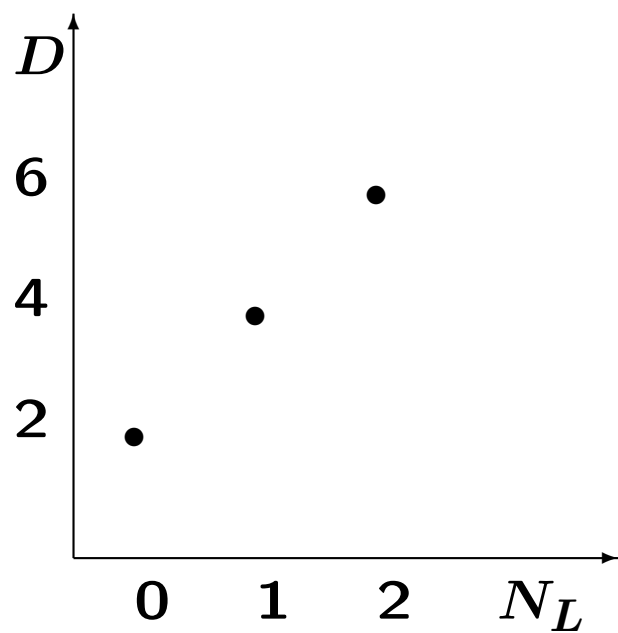
$$c_{1r} = c_1 + \delta c_1, \quad \delta c_1 = \frac{3mg_A^2}{128\pi^2 F^2} + \dots$$

Final result for the nucleon mass at order  $\mathcal{O}(q^3)$

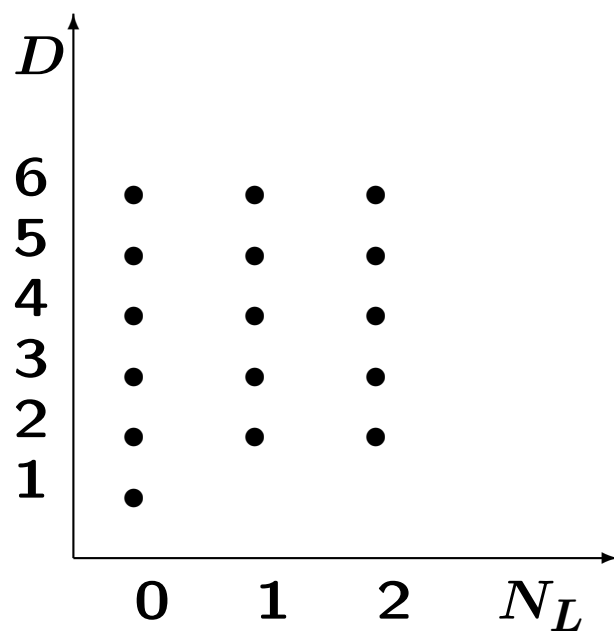
$$m_N = m - 4c_1 M^2 - \frac{3g_A^2 M^3}{32\pi F^2}$$

## Chiral versus loop expansion

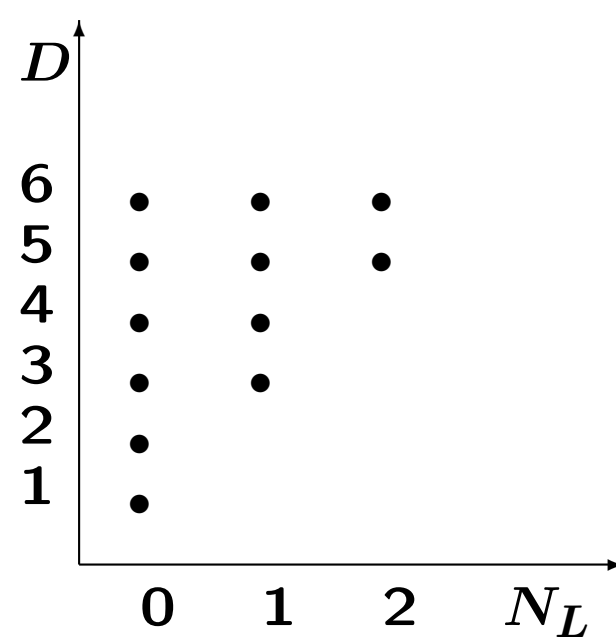
$\pi\pi: \overline{\text{MS}}$



$\pi N: \overline{\text{MS}}$



$\pi N: \text{EOMS, IR}$



## Applications

Mass of the nucleon at  $\mathcal{O}(q^3)$

- GSS ( $\widetilde{\text{MS}}$ )<sup>33</sup>

$$m_N = m - 4c_{1r}M^2 + \frac{3g_{Ar}^2 M^2}{32\pi^2 F_r^2} m - \frac{3g_{Ar}^2 M^3}{32\pi F_r^2}$$

Note:  $\widetilde{\text{MS}}$  implies infinite renormalization

$$c_1^0 = c_{1r} - \frac{3g_{Ar}^2}{128\pi^2} R$$

Solution to power counting problem

Term violating the power counting is analytic in small quantities and can thus be absorbed in counterterms

- EOMS<sup>34</sup>

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<sup>33</sup>J. Gasser, M. E. Sainio, A. Švarc, Nucl. Phys. B307, 779 (1988)

<sup>34</sup>T. Fuchs, J. Gegelia, G. Japaridze, S. Scherer, Phys. Rev. D 68, 056005 (2003)

**Rewrite**

$$c_{1r} = c_1 + \delta c_1, \quad \delta c_1 = \frac{3mg_A^2}{128\pi^2 F^2} + \dots$$

**Final result for the nucleon mass at order  $\mathcal{O}(q^3)$**

$$m_N = m - 4c_1 M^2 - \frac{3g_A^2 M^3}{32\pi F^2} + \mathcal{O}(M^4)$$

## Mass of the nucleon at $\mathcal{O}(q^4)$ <sup>35</sup>

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln(M/m) + k_4 M^4 + \mathcal{O}(M^5)$$

---

$$k_3 = \frac{3}{32\pi^2 F^2} \left( 8c_1 - c_2 - 4c_3 - \frac{g_A^2}{m} \right),$$

$$k_4 = \frac{3g_A^2}{32\pi^2 F^2 m} (1 + 4c_1 m) + \frac{3}{128\pi^2 F^2} c_2 - 16e_{38} - 2e_{115} - 2e_{116}.$$

---

$$m = [938.3 - 74.8 + 15.3 + 4.7 + 1.6 - 2.3] \text{ MeV} = 882.8 \text{ MeV}$$

$$\Delta m = 55.5 \text{ MeV}$$

Remark:  $m = m_N(m_u = 0, m_d = 0, m_s)$

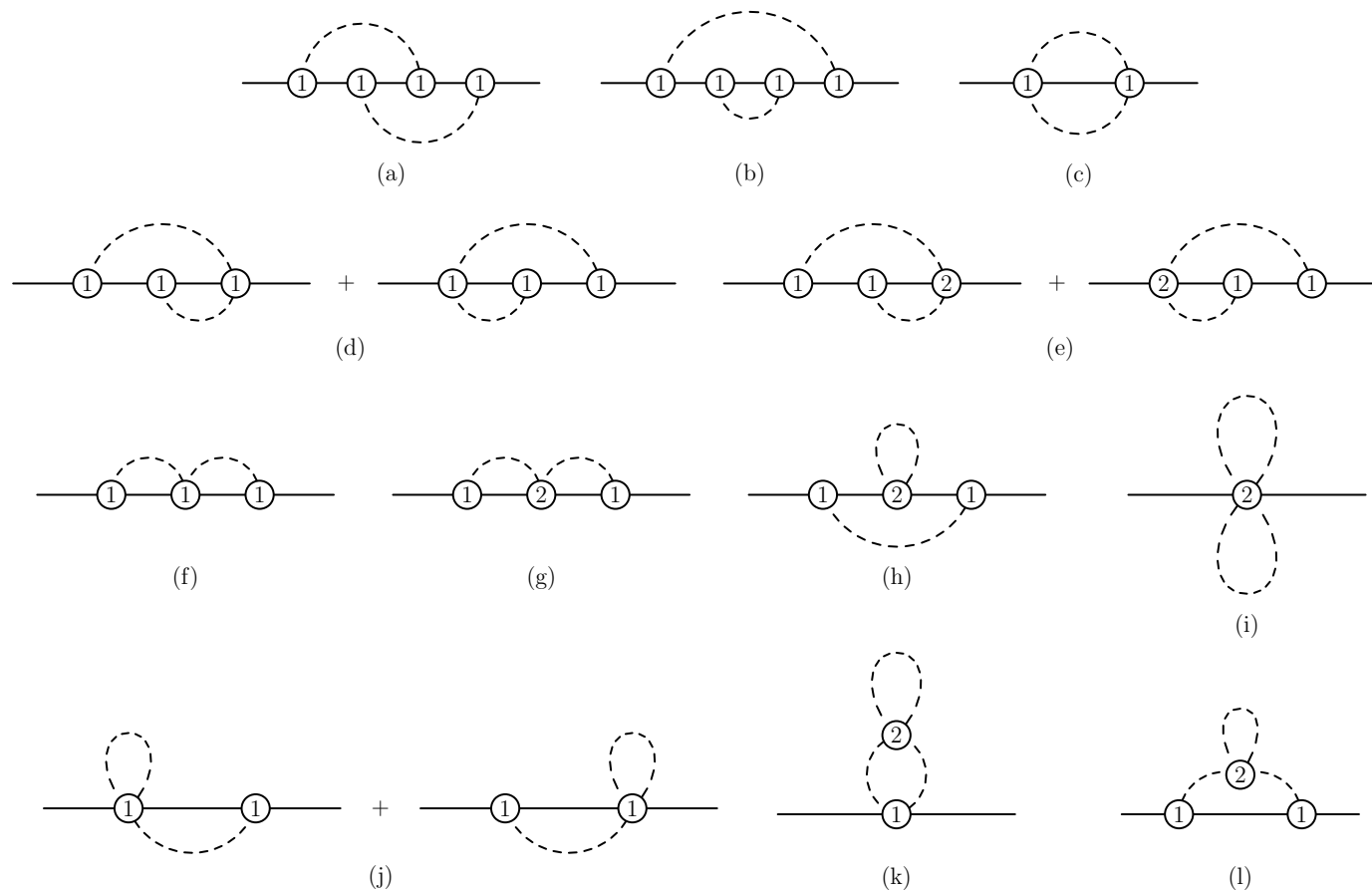
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<sup>35</sup>T. Becher and H. Leutwyler, Eur. Phys. J. C 9, 643 (1999); T. Fuchs, J. Gegelia, S. Scherer, Eur. Phys. J. A 19, 35 (2004)



# Mass of the nucleon at $\mathcal{O}(q^6)$ <sup>36</sup>

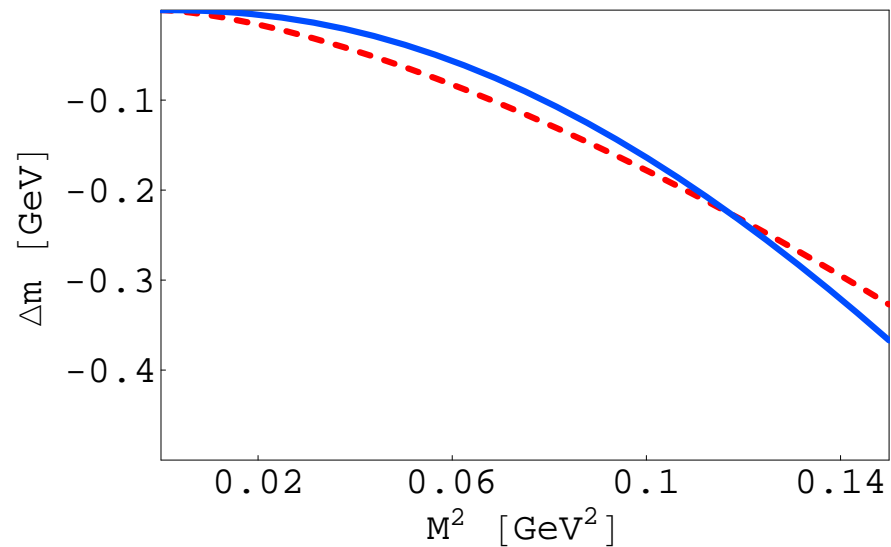
Two-loop contributions (M. R. Schindler, PhD thesis, 2007)



<sup>36</sup>M. R. Schindler, D. Djukanovic, J. Gegelia, S. S., Phys. Lett. B 649, 390 (2007); Nucl. Phys. A 803, 68 (2008)

$$\begin{aligned}
m_N = & m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M}{\mu} + k_4 M^4 \\
& + k_5 M^5 \ln \frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2 \frac{M}{\mu} + k_8 M^6 \ln \frac{M}{\mu} + k_9 M^6
\end{aligned}$$

} two loop



$M_0 \approx 360 \text{ MeV}$

(convergence)

At physical pion mass:  $-4.8 \text{ MeV} = 31\% \text{ of } k_2 M^3$

## Remarks

- Expressions of the coefficients in the chiral expansion of a physical quantity differ in various renormalization schemes
- However, the leading nonanalytic terms have to agree in all renormalization schemes
- Comparison with HBChPT <sup>37</sup>: Agreement for  $k_2$ ,  $k_3$ , and  $k_5$  (consistent!)

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<sup>37</sup> J. A. McGovern and M. C. Birse, Phys. Lett. B 446, 300 (1999)

$\sigma$  term <sup>38</sup>

**Definition of the so-called sigma commutator**

$$\sigma^{ab}(x) \equiv [Q_A^a(x_0), [Q_A^b(x_0), \mathcal{H}_{\text{sb}}(x)]], \quad a, b = 1, 2, 3$$

where

$$\mathcal{H}_{\text{sb}} = \bar{q}Mq = m_q(\bar{u}u + \bar{d}d)$$

**Measure of explicit symmetry breaking**

$$\sigma \equiv \frac{1}{2m_N} \langle p | \sigma^{11}(0) | p \rangle$$

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<sup>38</sup>T. Fuchs, J. Gegelia, S. Scherer, Eur. Phys. J. A 19, 35 (2004)

$$\sigma = \sigma_1 M^2 + \sigma_2 M^3 + \sigma_3 M^4 \ln\left(\frac{M}{m}\right) + \sigma_4 M^4 + O(M^5)$$

---

$$\sigma_1 = -4c_1$$

$$\sigma_2 = -\frac{9g_A^2}{64\pi F^2}$$

$$\sigma_3 = \frac{3}{16\pi^2 F^2} \left( 8c_1 - c_2 - 4c_3 - \frac{g_A^2}{m} \right)$$

$$\sigma_4 = \frac{3}{8\pi^2 F^2} \left[ \frac{3g_A^2}{8m} + c_1(1 + 2g_A^2) - \frac{c_3}{2} \right] + \alpha$$

---

$$\sigma = 45 \text{ MeV} = (74.8 - 22.9 - 9.4 - 2.0 + 4.5) \text{ MeV}$$

## Hellmann-Feynman theorem

Consider a Hermitian operator  $H(\lambda)$  with

$$H(\lambda)|\alpha(\lambda)\rangle = E(\lambda)|\alpha(\lambda)\rangle, \quad (1)$$

$$\langle\alpha(\lambda)|\alpha(\lambda)\rangle = 1. \quad (2)$$

Then

$$\frac{\partial E(\lambda)}{\partial\lambda} = \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle \quad (3)$$

Because

$$\begin{aligned} \frac{\partial E(\lambda)}{\partial\lambda} &\stackrel{(1),(2)}{=} \frac{\partial}{\partial\lambda} \langle\alpha(\lambda)|H(\lambda)|\alpha(\lambda)\rangle \\ &= \frac{\partial\langle\alpha(\lambda)|}{\partial\lambda} H(\lambda)|\alpha(\lambda)\rangle + \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle + \langle\alpha(\lambda)|H(\lambda)\frac{\partial|\alpha(\lambda)\rangle}{\partial\lambda} \\ &\stackrel{(1)}{=} E(\lambda)\frac{\partial}{\partial\lambda} \langle\alpha(\lambda)|\alpha(\lambda)\rangle + \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle \\ &\stackrel{(2)}{=} \langle\alpha(\lambda)|\frac{\partial H(\lambda)}{\partial\lambda}|\alpha(\lambda)\rangle \end{aligned}$$

Here: Multiply (3) with  $\lambda$  and identify

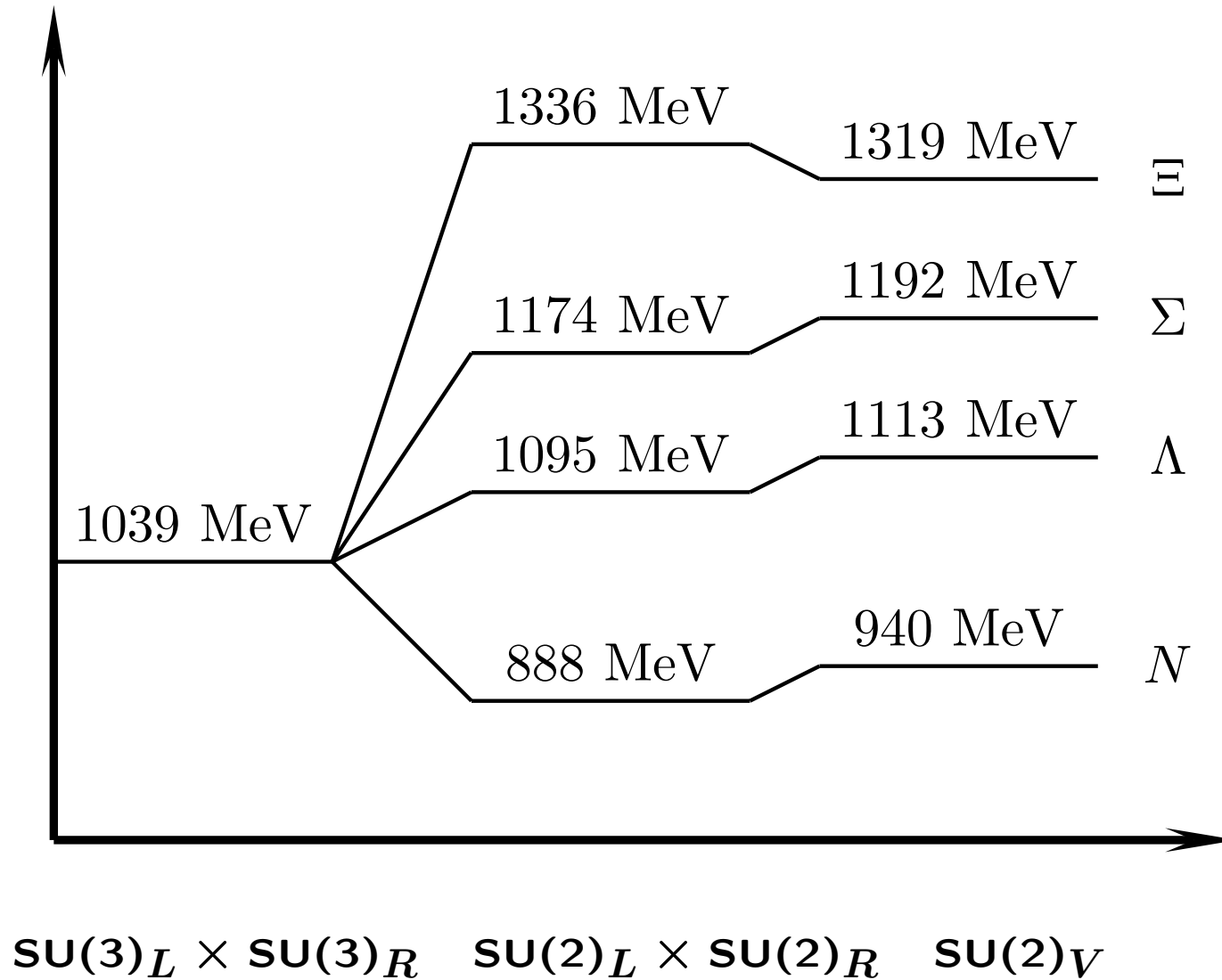
$$\begin{aligned}\lambda &\rightarrow \hat{m}, \\ |\alpha(\lambda)\rangle &\rightarrow |N(\hat{m})\rangle, \\ E(\lambda) &\rightarrow m_N(\hat{m}), \\ \frac{\partial H}{\partial \lambda} &\rightarrow \frac{\partial \mathcal{H}_{\text{QCD}}}{\partial \hat{m}} = \bar{u}u + \bar{d}d.\end{aligned}$$

Note that  $M^2 = 2B\hat{m}$  and thus

$$\sigma = M^2 \frac{\partial m_N}{\partial M^2}.$$

**Exercise:** Using the expressions for  $k_i$  and  $\sigma_i$  verify the Hellmann-Feynman theorem applied to the sigma term and the nucleon mass.

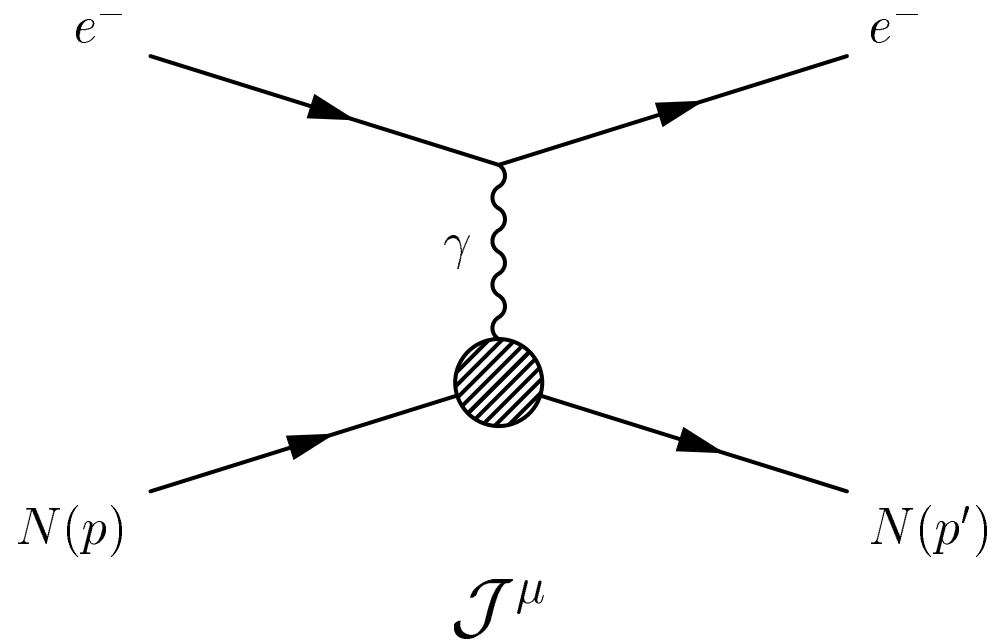
# Masses of the baryon octet at $\mathcal{O}(q^3)$ <sup>39</sup>



<sup>39</sup>B. C. Lehnhart, J. Gegelia, S. Scherer, J. Phys. G 31, 1 (2005)



## Electromagnetic form factors



## Electromagnetic current operator

$$\mathcal{J}^\mu(x) = \frac{2}{3} \bar{u}(x) \gamma^\mu u(x) - \frac{1}{3} \bar{d}(x) \gamma^\mu d(x) + \dots = \bar{q}(x) Q q(x) + \dots$$

## Definition of Dirac and Pauli form factors

$$\langle N(p') | \mathcal{J}^\mu(0) | N(p) \rangle = \bar{u}(p') \left[ \boxed{F_1^N(Q^2)} \gamma^\mu + i \frac{\sigma^{\mu\nu} q_\nu}{2m_p} \boxed{F_2^N(Q^2)} \right] u(p)$$

$$N = p, n, \quad q^\mu = p'^\mu - p^\mu, \quad Q^2 = -q^2$$

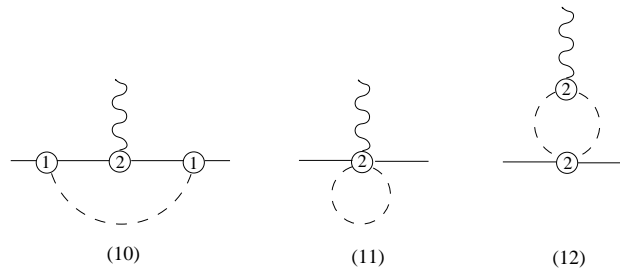
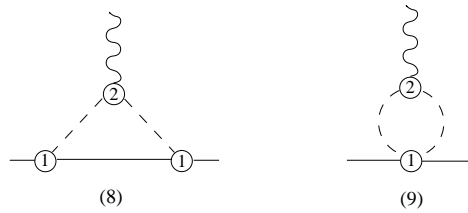
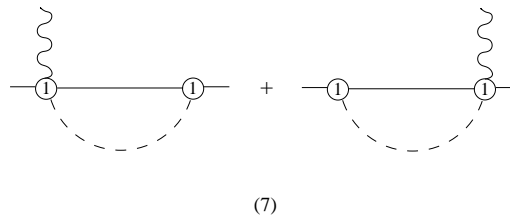
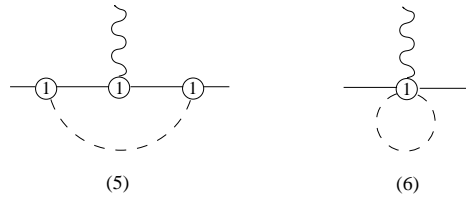
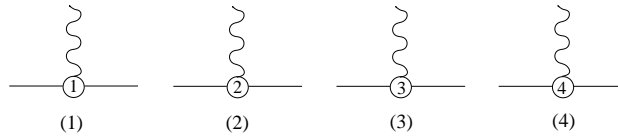
$$F_1^p(0) = 1, \quad F_1^n(0) = 0, \quad F_2^p(0) = 1.793, \quad F_2^n(0) = -1.913.$$

**Sachs form factors**

$$G_E^N(Q^2) = F_1^N(Q^2) - \frac{Q^2}{4m_N^2} F_2^N(Q^2)$$

$$G_M^N(Q^2) = F_1^N(Q^2) + F_2^N(Q^2)$$

# Diagrams at $\mathcal{O}(q^4)$



Diagrams potentially violating power counting: (5), (8), and (10).

EOMS subtractions

- Dirac form factor

$$\Delta F_1^{10} = \frac{g_A^2 m}{64\pi^2 F^2} (3c_7 - 2c_6\tau_3) t,$$

- Pauli form factor

$$\Delta F_2^5 = -\frac{g_A^2 m_N (m - 4c_1 M^2)}{32\pi^2 F^2} (3 - \tau_3),$$

$$\Delta F_2^8 = \frac{g_A^2 m_N (m - 4c_1 M^2)}{8\pi^2 F^2} \tau_3,$$

$$\Delta F_2^{10} = -\frac{g_A^2 m_N (m^2 - 8c_1 M^2 m)}{16\pi^2 F^2} (3c_7 - 2c_6\tau_3).$$

## Parameters

	$c_2$	$c_4$	$\tilde{c}_6$	$\tilde{c}_7$	$d_6$	$d_7$	$e_{54}$	$e_{74}$
<b>EOMS</b>	<b>2.66</b>	<b>2.45</b>	<b>1.26</b>	<b>-0.13</b>	<b>-0.57</b>	<b>-0.44</b>	<b>0.27</b>	<b>1.71</b>
<b>IR</b>	<b>2.66</b>	<b>2.45</b>	<b>0.47</b>	<b>-1.87</b>	<b>0.32</b>	<b>-0.89</b>	<b>0.33</b>	<b>1.65</b>

The LECs  $c_i$  are given in units of  $\text{GeV}^{-1}$ , the  $d_i$  in units of  $\text{GeV}^{-2}$ , and the  $e_i$  in units of  $\text{GeV}^{-3}$ .

$c_2$  and  $c_4$  from  $\pi N$  scattering;

$\tilde{c}_6$  and  $\tilde{c}_7$  from anomalous magnetic moments;

$d_6$ ,  $d_7$ ,  $e_{54}$ , and  $e_{74}$  from charge and magnetic radii: <sup>40</sup>

$$r_E^p = 0.848 \text{ fm},$$

$$r_M^p = 0.857 \text{ fm},$$

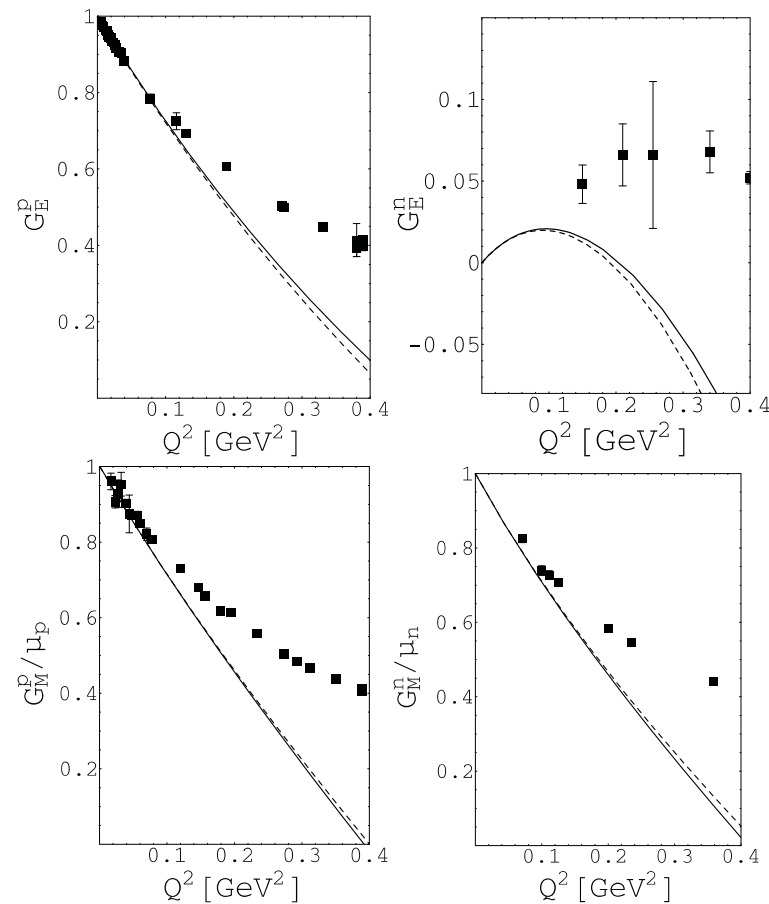
$$r_E^n = 0.113 \text{ fm},$$

$$r_M^n = 0.879 \text{ fm}.$$

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<sup>40</sup>H. W. Hammer and U.-G. Meißner, Eur. Phys. J. A 20, 469 (2004)

# Sachs form factors <sup>41</sup> (T. Fuchs, PhD thesis, 2003)



to be continued

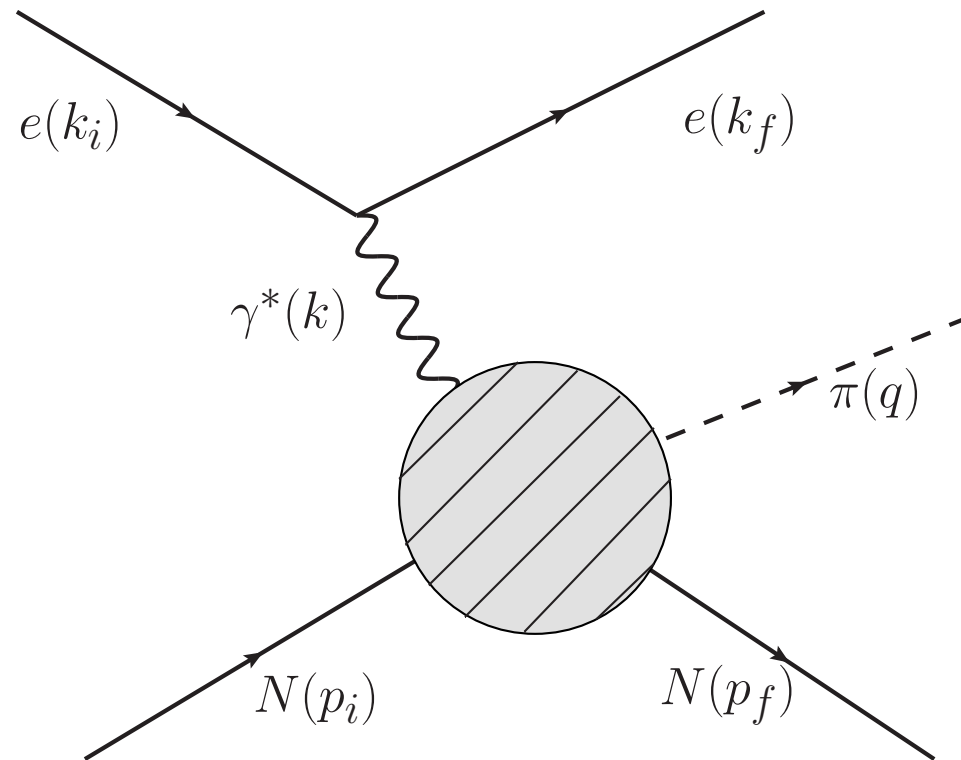
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<sup>41</sup>B. Kubis and U.-G. Meißner, Nucl. Phys. A679, 698 (2001); T. Fuchs, J. Gegelia, S. Scherer, J. Phys. G 30, 1407 (2004); data taken from J. Friedrich and Th. Walcher, Eur. Phys. J. A 17, 607 (2003)

# Pion Photo- and Electroproduction at $\mathcal{O}(q^4)$ with $\chi$ MAID <sup>42</sup>

$$e(k_i) + N(p_i) \rightarrow e(k_f) + N(p_f) + \pi(q)$$

(B. C. Lehnhart, PhD thesis 2007, M. Hilt, PhD thesis 2011)



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<sup>42</sup>M. Hilt, S. Scherer, L. Tiator, Phys. Rev. C 87, 045204 (2013), M. Hilt, B. C. Lehnhart, S. Scherer, L. Tiator, Phys. Rev. C 88, 055207 (2013)

## One-photon-exchange approximation

$\mathcal{M}$  = leptonic vertex  $\times$   $i$  propagator  $\times$  hadronic vertex

$$= \epsilon_\mu \mathcal{M}^\mu,$$

$$\epsilon_\mu = e \frac{\bar{u}(k_f) \gamma_\mu u(k_i)}{k^2},$$

$$\mathcal{M}^\mu = -ie \langle N(p_f), \pi(q) | J^\mu(0) | N(p_i) \rangle.$$

## Current conservation

$$k_\mu \mathcal{M}^\mu = 0$$

## Parameterization in terms of invariant amplitudes

$$\mathcal{M}^\mu = \bar{u}(p_f) \left( \sum_{i=1}^6 A_i M_i^\mu \right) u(p_i),$$



with

$$\begin{aligned}M_1^\mu &= -\frac{i}{2}\gamma_5 (\gamma^\mu \not{k} - \not{k}\gamma^\mu), \\M_2^\mu &= 2i\gamma_5 \left[ P^\mu k \cdot \left( q - \frac{1}{2}k \right) - \left( q^\mu - \frac{1}{2}k^\mu \right) k \cdot P \right], \\M_3^\mu &= -i\gamma_5 (\gamma^\mu k \cdot q - \not{k}q^\mu), \\M_4^\mu &= -2i\gamma_5 (\gamma^\mu k \cdot P - \not{k}P^\mu) - 2m_N M_1^\mu, \\M_5^\mu &= i\gamma_5 (k^\mu k \cdot q - q^\mu k^2), \\M_6^\mu &= -i\gamma_5 (\not{k}k^\mu - \gamma^\mu k^2).\end{aligned}$$

**Isospin decomposition: Four physical channels**

$$\begin{aligned}A_i(\gamma^{(*)}p \rightarrow n\pi^+) &= \sqrt{2} \left( A_i^{(-)} + A_i^{(0)} \right), \\A_i(\gamma^{(*)}p \rightarrow p\pi^0) &= A_i^{(+)} + A_i^{(0)}, \\A_i(\gamma^{(*)}n \rightarrow p\pi^-) &= -\sqrt{2} \left( A_i^{(-)} - A_i^{(0)} \right), \\A_i(\gamma^{(*)}n \rightarrow n\pi^0) &= A_i^{(+)} - A_i^{(0)}.\end{aligned}$$

expressed in terms of three isospin amplitudes (0), (+), and (-)

1.  $\mathcal{O}(q^4)$ : 20 tree-level diagrams + 85 loop diagrams
2. Calculate loop contributions numerically using CAS MATHEMATICA with FeynCalc and LoopTools packages
3. LECs from other processes

LEC	Source
$l_3$	$M_\pi = 134.977$ MeV
$l_4, l_6$	pion form factor
$c_1$	proton mass $m_p = 938.272$ MeV
$c_2, c_3, c_4$	pion-nucleon scattering
$c_6, c_7$	magnetic moment of proton ( $\mu_p = 2.793$ ) and neutron ( $\mu_n = -1.913$ )
$d_6, d_7, e_{54}, e_{74}$	world data for nucleon electromagnetic form factors ( $Q^2 < 0.3$ GeV <sup>2</sup> )
$d_{16}$	axial-vector coupling constant $g_A = 1.2695$
$d_{18}$	pion-nucleon coupling
$d_{22}$	axial radius of the nucleon $\langle r_A^2 \rangle = 12/M_A^2$ , $M_A = 1.026$ GeV

4. Checks: Current conservation and crossing symmetry

5. Analytic expressions for the contact diagrams

(a) 4 LECs at  $\mathcal{O}(q^3)$

(b) 15 LECs at  $\mathcal{O}(q^4)$

6. Web interface  $\chi$ MAID

[<http://www.kph.uni-mainz.de/MAID/chiralmaid/>]

7. Fits to available experimental data



[ChiralMAID info and updates \(please read first\)](#)

## Pion Photo- and Electroproduction on the Nucleon in relativistic chiral perturbation theory

[M. Hilt](#), [S. Scherer](#), [L. Tiator](#)

- [Electromagnetic Multipoles](#) ( $E_{l\pm}$ ,  $M_{l\pm}$ ,  $L_{l\pm}$ ,  $S_{l\pm}$ )
- [Amplitudes](#) ( $F_1, \dots, F_6$ ,  $H_1, \dots, H_6$ ,  $A_1, \dots, A_6$ )
- [Differential Cross Sections](#) ( $ds_T$ ,  $ds_L$ ,  $ds_{LT}$ ,  $ds_{TT}$ , ...)
- [5-fold Diff. Cross Section](#) ( $d^5s$ ,  $G$ ,  $ds^V = ds_T + e ds_L + e ds_{TT} \cos 2\theta + \dots$ )
- [Total Cross Sections](#) ( $s_T$ ,  $s_L$ ,  $s_{LT}$ ,  $s_{TT}$ , ...)
- [Transverse Polarization Observables](#) ( $ds/dW$ ,  $T$ ,  $S$ ,  $P$ ,  $E$ ,  $F$ ,  $G$ ,  $H$ , ...)

External services:

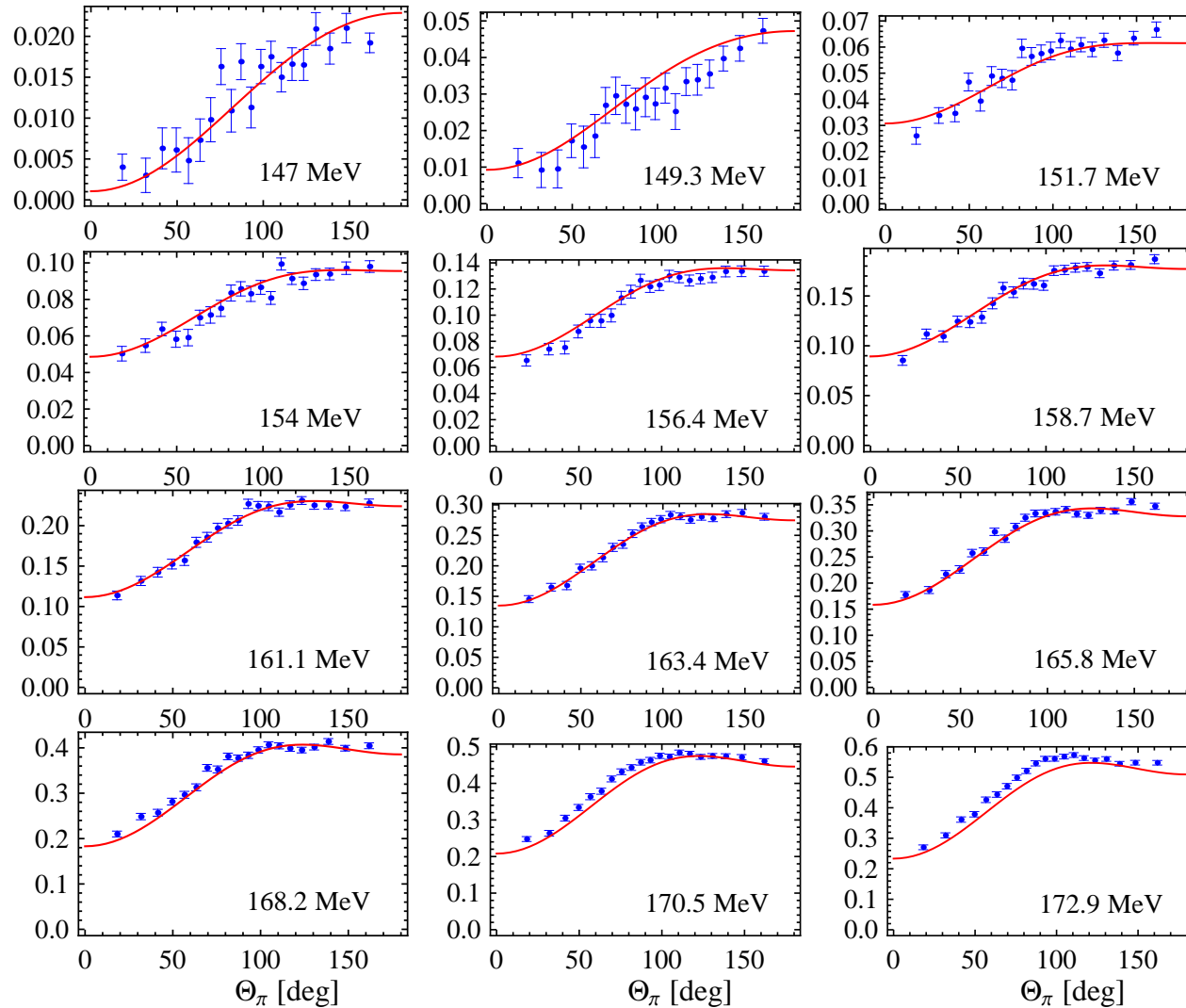
[MAID Homepage](#) [MAID2003](#) [DMT2001](#) [KAON-MAID](#) [ETA-MAID2000](#) [ETA-MAID2003](#) [ETA'-MAID](#)

[A1 kinematics calculator for electroproduction \(Java\)](#)

[SAID Partial-Wave Analyses](#)

[Back to Theory Group Homepage](#)

# Angular distribution for $d\sigma/d\Omega_\pi$ in $\mu\text{b}/\text{sr}$ for $\gamma + p \rightarrow p + \pi^0$ <sup>43</sup>



<sup>43</sup>Data taken from D. Hornidge et al., Phys. Rev. Lett. 111, 062004 (2013)

## Data available for

1.  $\gamma + p \rightarrow p + \pi^0$

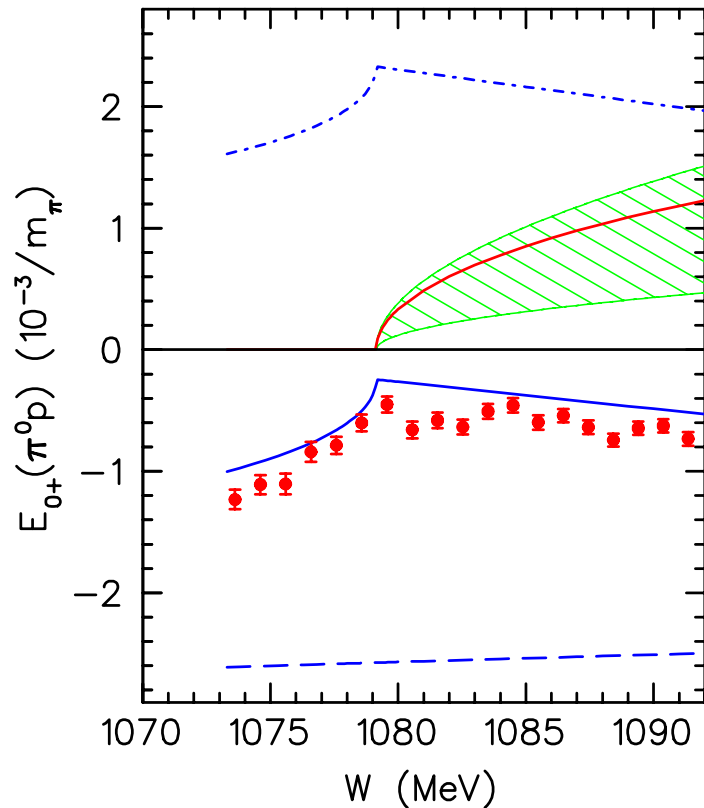
2.  $\gamma^* + p \rightarrow p + \pi^0$

3.  $\gamma + p \rightarrow n + \pi^+$  and  $\gamma + n \rightarrow p + \pi^-$

4.  $\gamma^{(*)} + p \rightarrow n + \pi^+$

Isospin channel	LEC	Value
0	$d_9$ [GeV <sup>-2</sup> ]	$-1.22 \pm 0.12$
0	$e_{48}$ [GeV <sup>-3</sup> ]	$5.2 \pm 1.4$
0	$e_{49}$ [GeV <sup>-3</sup> ]	$0.9 \pm 2.6$
0	$e_{50}$ [GeV <sup>-3</sup> ]	$2.2 \pm 0.8$
0	$e_{51}$ [GeV <sup>-3</sup> ]	$6.6 \pm 3.6$
0	$e_{52}^*$ [GeV <sup>-3</sup> ]	$-4.1$
0	$e_{53}^*$ [GeV <sup>-3</sup> ]	$-2.7$
0	$e_{112}$ [GeV <sup>-3</sup> ]	$9.3 \pm 1.6$
+	$d_8$ [GeV <sup>-2</sup> ]	$-1.09 \pm 0.12$
+	$e_{67}$ [GeV <sup>-3</sup> ]	$-8.3 \pm 1.5$
+	$e_{68}$ [GeV <sup>-3</sup> ]	$-0.9 \pm 2.6$
+	$e_{69}$ [GeV <sup>-3</sup> ]	$-1.0 \pm 2.2$
+	$e_{71}$ [GeV <sup>-3</sup> ]	$-4.4 \pm 3.7$
+	$e_{72}^*$ [GeV <sup>-3</sup> ]	$10.5$
+	$e_{73}^*$ [GeV <sup>-3</sup> ]	$2.1$
+	$e_{113}$ [GeV <sup>-3</sup> ]	$-13.7 \pm 2.6$
-	$d_{20}$ [GeV <sup>-2</sup> ]	$4.34 \pm 0.08$
-	$d_{21}$ [GeV <sup>-2</sup> ]	$-3.1 \pm 0.1$
-	$e_{70}$ [GeV <sup>-3</sup> ]	$3.9 \pm 0.3$

**Example:**  $\gamma + p \rightarrow p + \pi^0$  at  $\mathcal{O}(q^3)$  <sup>44</sup>

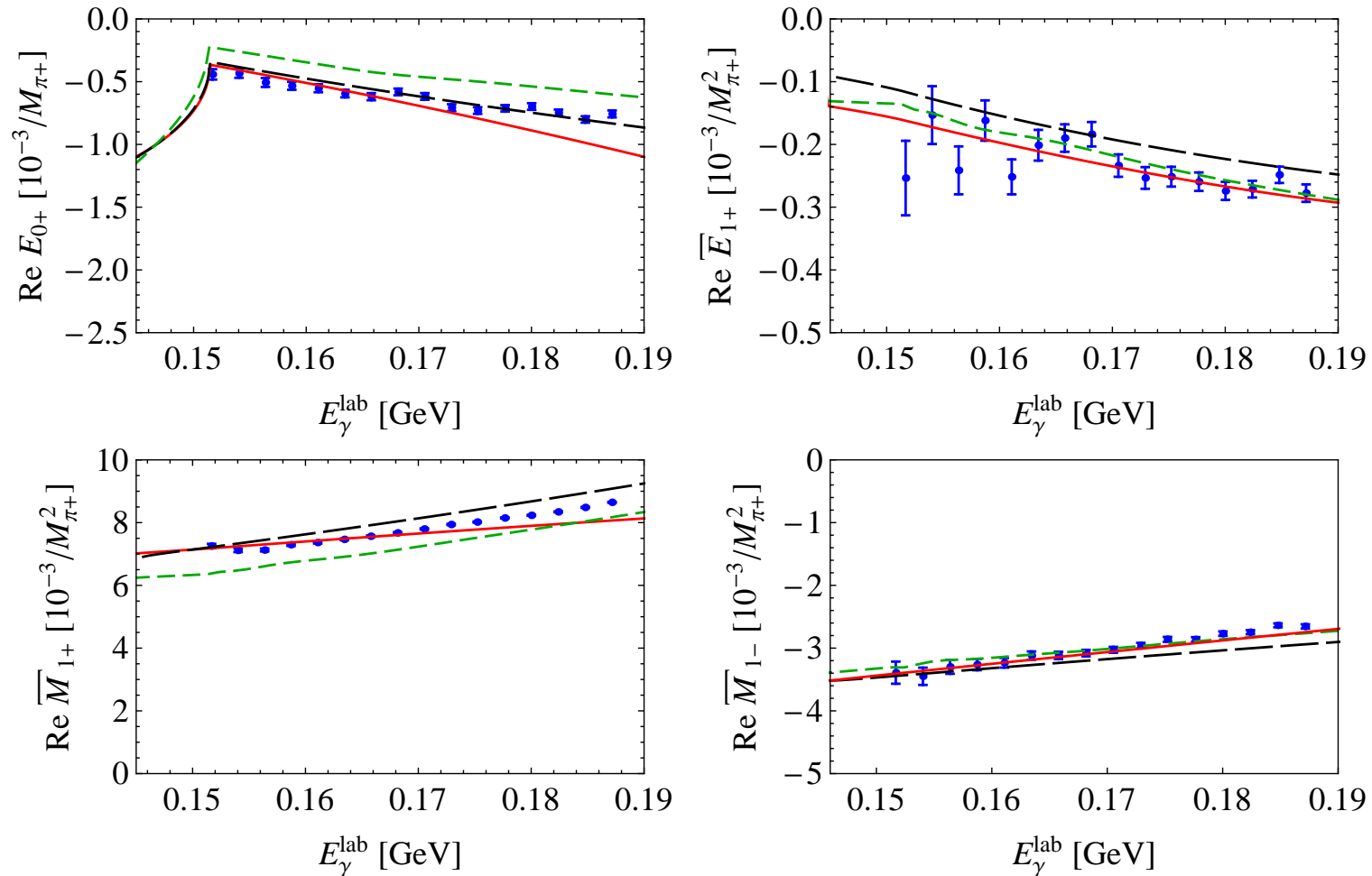


**Solid blue line:** real part  
**Solid red line:** imaginary part  
**Cusp** from taking  $m_n$  and  $m_{\pi^+}$  in loop  
**Long-dashed blue line:** tree-level contribution  
**Dashed-dotted blue line:** loop contribution  
**Green band:** Imaginary part from ansatz  $\text{Im}(E_{0+}) = \beta|\vec{q}|$

<sup>44</sup>Data taken from A. Schmidt et al., Phys. Rev. Lett. 87, 232501 (2001)

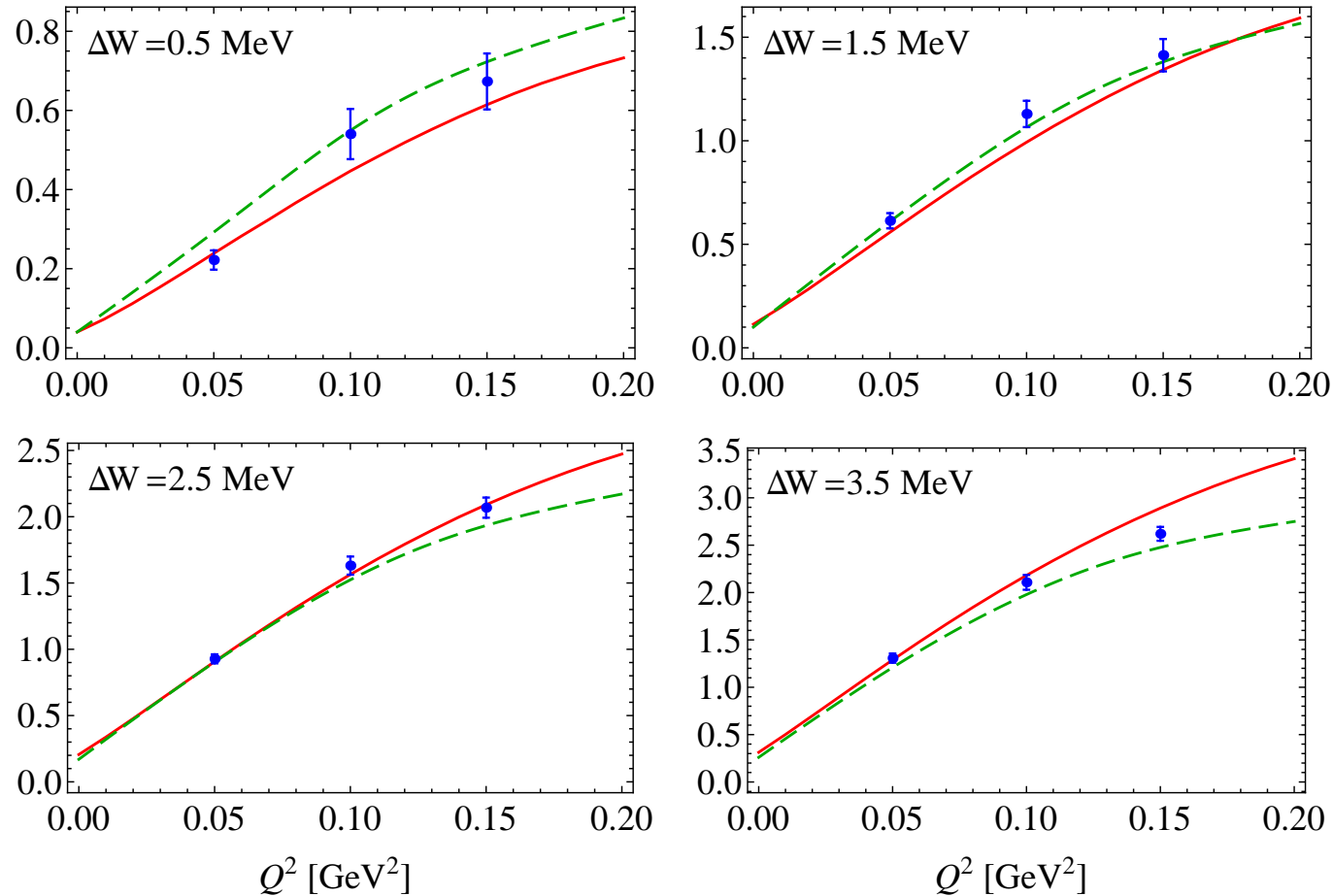


**$S$ - and reduced  $P$ -wave multipoles for  $\gamma + p \rightarrow p + \pi^0$ .**



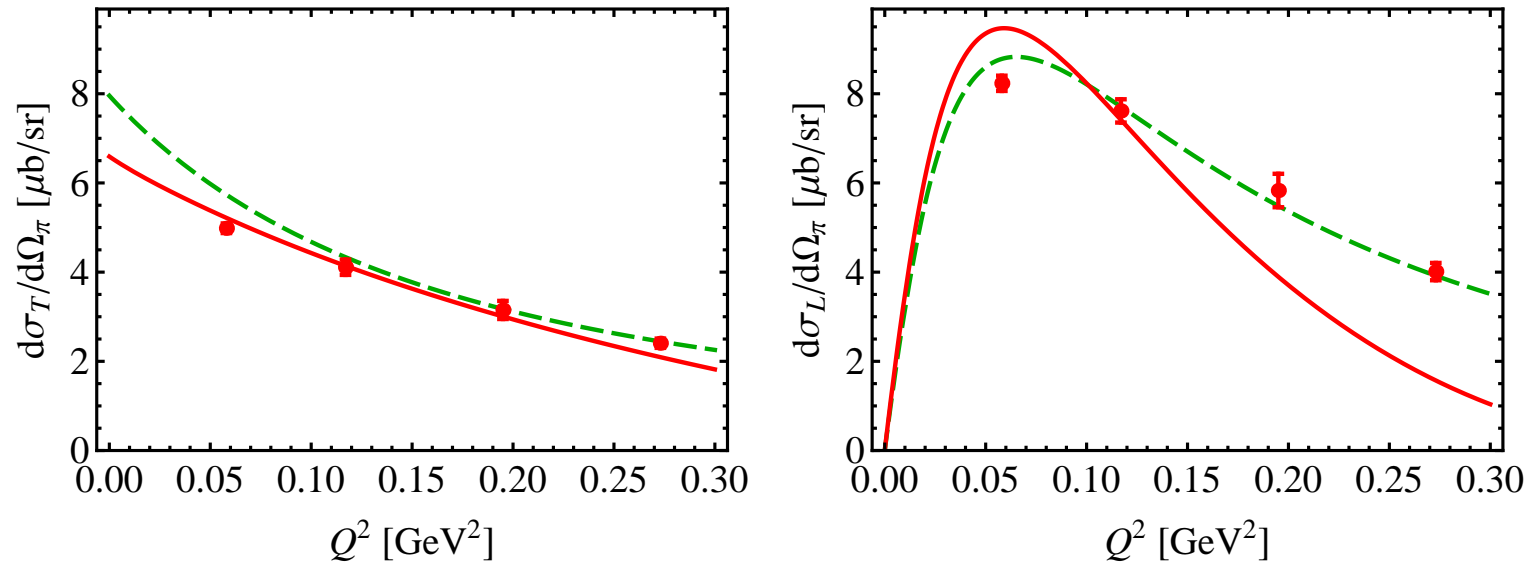
**Red curves: RChPT; green curves: DMT model;  
black curves: Gasparyan & Lutz; data from Hornidge et al. (2012)**

Total cross sections for  $\gamma^* + p \rightarrow p + \pi^0$  in  $\mu\text{b}$  as a function of  $Q^2$  for different cm energies above threshold  $\Delta W$  in MeV.



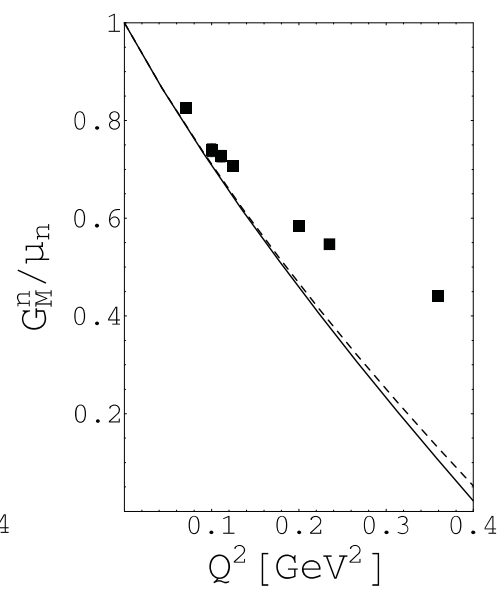
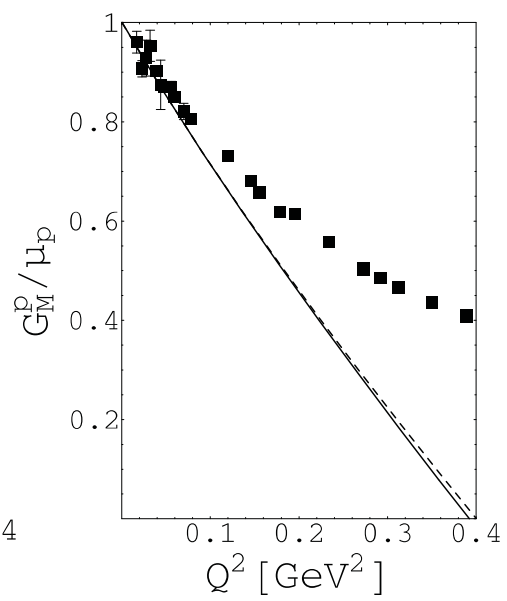
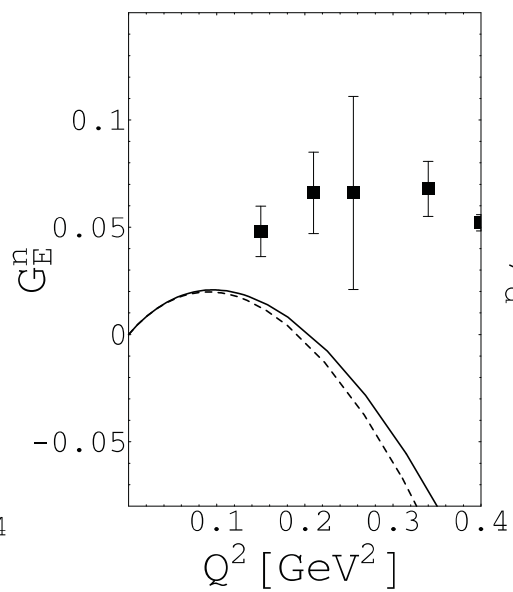
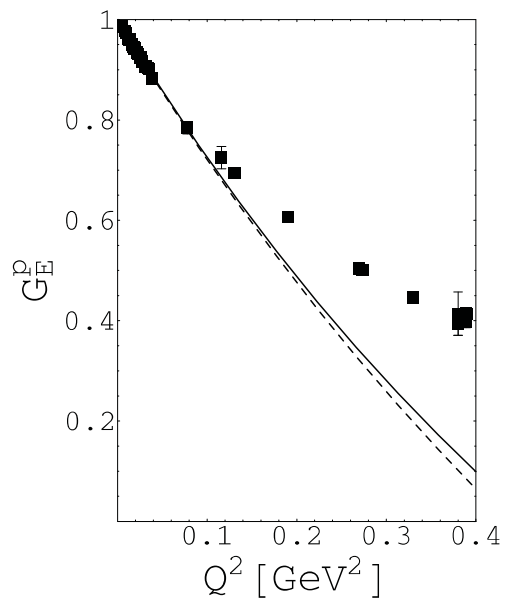
Red curves: RChPT; green curves: DMT model;  
data from Merkel et al. 2009, 2011

Differential cross sections as a function of  $Q^2$  for  $\gamma^* + p \rightarrow n + \pi^+$  at  $W = 1125$  MeV and  $\Theta_\pi = 0^\circ$ .



Red curves: RChPT; green curves: DMT model;  
 data from Baumann (PhD thesis, JGU, 2005) **Chiral Effective  
 Field Theory (Including Resonances, Constraints, ...) Electroma-  
 gnetic form factors revisited I** <sup>45</sup>

<sup>45</sup>B. Kubis and U.-G. Meißner, Nucl. Phys. A679, 698 (2001);  
 T. Fuchs, J. Gegelia, S. Scherer, J. Phys. G 30, 1407 (2004); data taken from  
 J. Friedrich and Th. Walcher, Eur. Phys. J. A 17, 607 (2003)



**Vector meson dominance model → Important contributions to the electromagnetic form factors** <sup>46</sup>

**In standard ChPT: Vector meson contributions in low-energy constants**

$$\frac{1}{q^2 - M_V^2} = -\frac{1}{M_V^2} \left[ 1 + \frac{q^2}{M_V^2} + \left( \frac{q^2}{M_V^2} \right)^2 + \mathcal{O}(q^6) \right]$$

**Inclusion of vector mesons ⇒ re-summation of higher-order contributions**

**Reformulated IR regularization and EOMS scheme allow for consistent inclusion of vector mesons**

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<sup>46</sup>**B. Kubis and U.-G. Meißner, Nucl. Phys. A679, 698 (2001)**

## Additional Lagrangians <sup>47</sup>

$$\mathcal{L}_{\pi V}^{(3)} = -f_\rho \text{Tr}(\rho^{\mu\nu} f_{\mu\nu}^+) - f_\omega \omega^{\mu\nu} f_{\mu\nu}^{(s)} - f_\phi \phi^{\mu\nu} f_{\mu\nu}^{(s)} + \dots$$

$$V_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu, \quad V = \rho, \omega, \phi, \quad \nabla_\mu V_\nu = \partial_\mu V_\nu + [\Gamma_\mu, V_\nu]$$

$$\mathcal{L}_{NV}^{(0)} = \frac{1}{2} \sum_{V=\rho,\omega,\phi} g_V \bar{\Psi} \gamma^\mu V_\mu \Psi$$

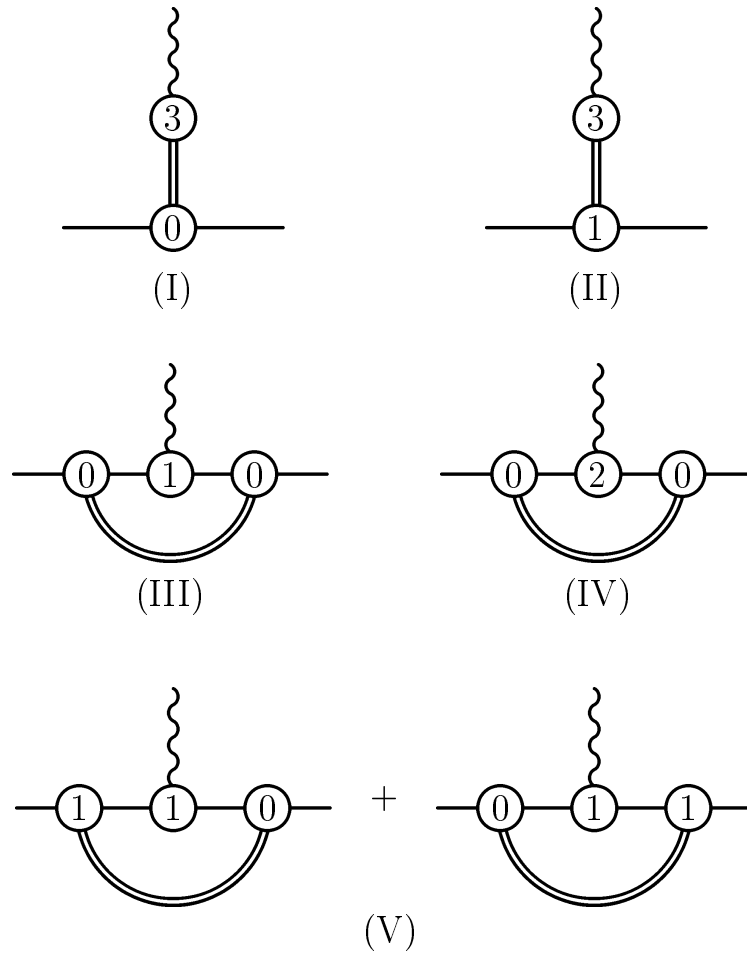
$$\mathcal{L}_{NV}^{(1)} = \frac{1}{4} \sum_{V=\rho,\omega,\phi} G_V \bar{\Psi} \sigma^{\mu\nu} V_{\mu\nu} \Psi$$

Additional rules:

- Vector meson propagator  $\sim \mathcal{O}(q^0)$
- Vertex from  $\mathcal{L}_V^{(i)} \sim \mathcal{O}(q^i)$

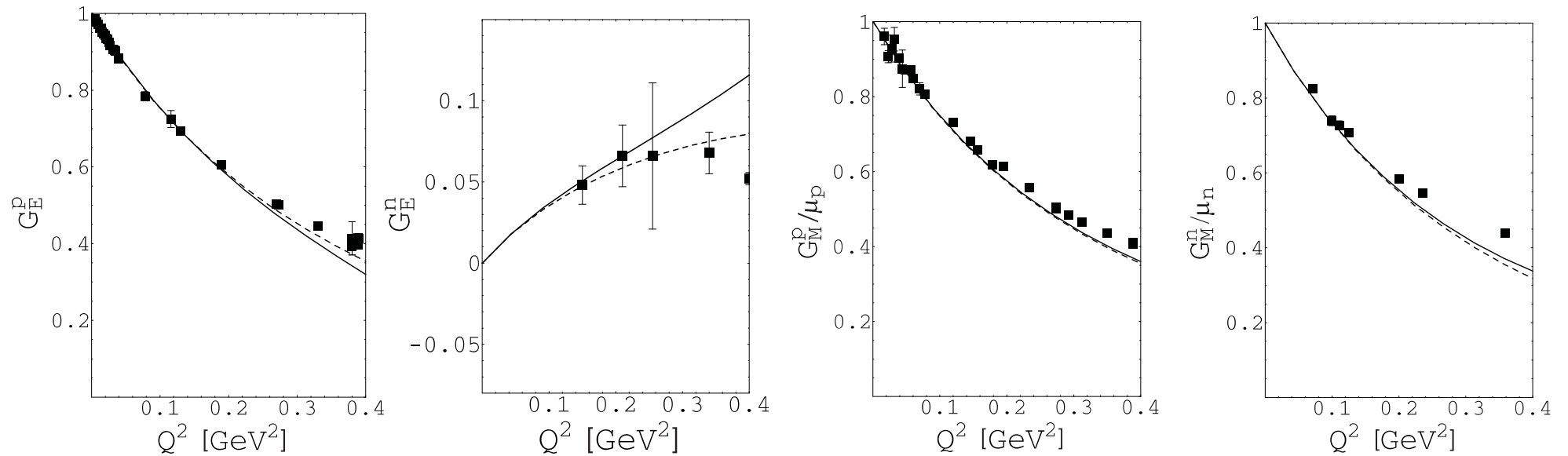
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<sup>47</sup>G. Ecker, J. Gasser, H. Leutwyler, A. Pich, E. de Rafael, Phys. Lett. B 223, 425 (1989)



**Feynman diagrams involving vector mesons contributing to the electromagnetic form factors up to and including  $\mathcal{O}(q^4)$**

## E.m. form factors including vector mesons at $\mathcal{O}(q^4)$ <sup>48</sup>

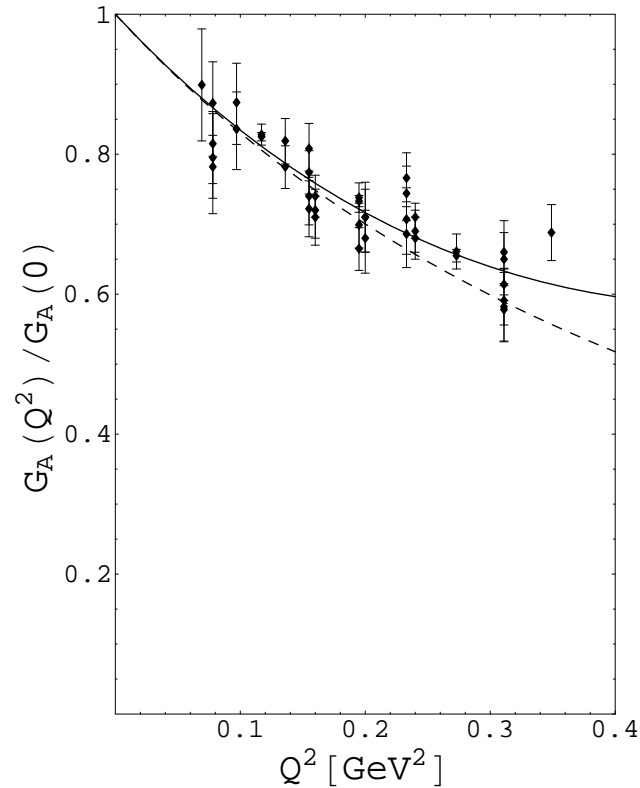
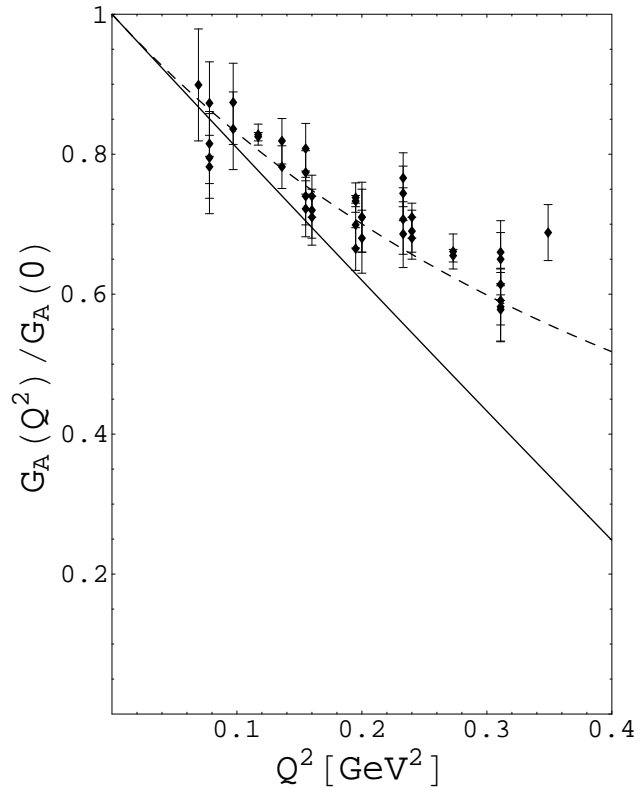


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<sup>48</sup>M. R. Schindler, J. Gegelia, S. Scherer, *Eur. Phys. J. A* 26, 1 (2005); data taken from J. Friedrich and Th. Walcher, *Eur. Phys. J. A* 17, 607 (2003)



# Axial form factor $G_A$ including the $a_1$ meson at $\mathcal{O}(q^4)$ <sup>49</sup>



**Solid line: ChPT**

**Dashed line: ChEFT**

**Solid line: Dipole fit**

**Dashed line: ChEFT**

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<sup>49</sup>M. R. Schindler, T. Fuchs, J. Gegelia, S. Scherer, Phys. Rev. C 75, 025202 (2007)

## Perturbative calculations in effective field theory

1. Write down most general Lagrangian
2. Draw all diagrams contributing to a given process:  
Tree-level diagrams, loop diagrams  $\rightsquigarrow$  ultraviolet divergences, regularization (of infinities)
3. Subtract diagrams and specify renormalization condition
4. Define power counting for renormalized (subtracted) diagrams
5. Sum all subtracted diagrams to a given order
6. Remove regularization

## Effective field theories including constraints

- **Most general Lagrangian (discussion of vector fields)** <sup>50</sup>

$$\mathcal{L}_{V\text{eff}} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C$$

- $\mathcal{L}_A$ : Finite # of interaction terms with dimensionless coupling constants
  - $\mathcal{L}_B$ : Infinite # of “non-renormalizable” interactions
  - $\mathcal{L}_C$ : Interaction with other d.o.f.
- **Assumption**
    - Dimensionless couplings “small”
    - “Non-renormalizable” interactions suppressed by powers of some scale

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<sup>50</sup>D. Djukanovic, J. Gegelia, S. Scherer, Int. J. Mod. Phys. A 25, 3603 (2010)

- **Input (symmetries)**

- 3 interacting massive vector fields
- Lorentz invariance + parity conservation

- **Structure of  $\mathcal{L}_A$**

$$\mathcal{L}_A = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$$

- **Free Lagrangian  $\mathcal{L}_2$**

$$\mathcal{L}_2 = -\frac{1}{4} V_{\mu\nu}^a V^{a\mu\nu} + \frac{M_a^2}{2} V_\mu^a V^{a\mu}, \quad V_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a, \quad a = 1, 2, 3$$

- Invariance under global U(1) (charge conservation)

$$V_\mu^a \rightarrow V_\mu^a - \varepsilon \epsilon^{3ab} V_\mu^b \quad \Rightarrow \quad M_1 = M_2 = M$$

• Interaction Lagrangian  $\mathcal{L}_3$

$$\mathcal{L}_3 = -g^{abc} V_\mu^a V_\nu^b \partial^\mu V^{c\nu}$$

–  $g^{abc}$ :  $3^3 = 27$  real coupling constants (hermiticity)

– U(1) invariance  $\Rightarrow$

$$(g^{abd} \epsilon^{3dc} + g^{adc} \epsilon^{3db} + g^{dbc} \epsilon^{3da}) V_\mu^a V_\nu^b \partial^\mu V^{c\nu} = 0$$

– Parameterize couplings in terms of 7 real parameters

$$\begin{aligned} g^{333} &= g_1, & g^{113} &= g_2, & g^{123} &= -g_3, & g^{213} &= g_3, \\ g^{223} &= g_2, & g^{311} &= g_4, & g^{321} &= -g_5, & g^{312} &= g_5, \\ g^{322} &= g_4, & g^{131} &= g_6, & g^{231} &= -g_7, & g^{132} &= g_7, \\ g^{232} &= g_6 \end{aligned}$$

Remaining 14 constants vanish

– Structures of the type  $\partial_\mu V^{a\mu} V_\nu^b V^{c\nu}$  not independent (total derivative)

– Invariance under charge conjugation  $V_\mu^a \mapsto (-)^a V_\mu^a \Rightarrow$

$$(-)^{a+b+c} = 1 \quad \text{or} \quad a + b + c \text{ even}$$

Would eliminate  $g_1, g_2, g_4,$  and  $g_6 \Rightarrow 3$  real coupling constants

– SU(2) symmetry  $\Rightarrow$

$$g^{abc} = \epsilon^{abc} g, \quad \text{also} \quad M_3 = M$$

#### ● Interaction Lagrangian $\mathcal{L}_4$

$$\mathcal{L}_4 = -h^{abcd} V_\mu^a V_\nu^b V^{c\mu} V^{d\nu}$$

–  $h^{abcd}$ : 21 real coupling constants (permutation symmetry + hermiticity)

– Invariance under U(1)  $\Rightarrow 5$  real parameters

$$h^{1111} = \frac{d_1 + d_2}{4}, \dots, h^{3333} = d_5$$

- In this case, charge conservation also implies charge conjugation invariance, i.e.,  $a + b + c + d$  even
- SU(2) symmetry  $\Rightarrow$  2 parameters
- Summary of # of parameters of Lagrangian  $\mathcal{L}_A$ 
  - No internal symmetry: 3 masses, 27 coupling constants  $g$ , 21 coupling constants  $h \Rightarrow$  51 real parameters
  - U(1) symmetry: 2 masses, 7 coupling constants  $g$ , 5 coupling constants  $h \Rightarrow$  14 real parameters
  - U(1) symmetry + charge conjugation: 2 masses, 3 coupling constants  $g$ , 5 coupling constants  $h \Rightarrow$  10 real parameters
  - SU(2) symmetry: 1 mass, 1 coupling constant  $g$ , 2 coupling constants  $h \Rightarrow$  4 real parameters

● **Hamiltonian method** <sup>51</sup>

Goal: Consistent description of three interacting spin-1 d.o.f.

Hamiltonian framework

- $3 \times [3 \text{ (fields)} + 3 \text{ (momenta)}] = 18 \text{ d.o.f.}$
- However  $3 \times [4 \text{ (fields)} + 4 \text{ (momenta)}] = 24 \text{ d.o.f.}$
- $\Rightarrow$  consistent theory requires **6 constraints**

Conjugated momenta

$$\pi_0^a = \frac{\partial \mathcal{L}_V}{\partial \dot{V}_0^a} = -g^{bca} V_0^b V_0^c$$
$$\pi_i^a = \frac{\partial \mathcal{L}_V}{\partial \dot{V}_i^a} = V_{0i}^a + g^{bca} V_0^b V_i^c$$

---

<sup>51</sup> P. A. M. Dirac, *Lectures on Quantum Mechanics* (Dover, Mineola, New York, 2001); D. M. Gitman and I. V. Tyutin, *Canonical Quantization of Fields with Constraints* (Nauka, Moscow, 1986; extended English version: Springer, Berlin, 1990)



Transition  $\mathcal{L}$  to  $\mathcal{H} \Rightarrow$  Solve

$$\dot{V}_i^a = \pi_i^a + \partial_i V_0^a - g^{bca} V_0^b V_i^c$$

But velocities  $\dot{V}_0^a$  cannot be solved  $\Rightarrow$  **3 primary constraints**

$$\phi_1^a = \pi_0^a + g^{bca} V_0^b V_0^c \approx 0$$

Total Hamiltonian density:

$$\mathcal{H}_1 = \phi_1^a z^a + \mathcal{H}$$

where

$$\mathcal{H} = \frac{\pi_i^a \pi_i^a}{2} + \dots + h^{abcd} V_\mu^a V_\nu^b V^{c\mu} V^{d\nu}$$

$z^a$ : Arbitrary functions (Lagrange multipliers); to be determined

Primary constraints have to be **conserved in time**

$$\{\phi_1^a, H_1\} \stackrel{\text{work}}{=} A^{ab} z^b + \chi^a \approx 0, \quad a = 1, 2, 3$$

– System of 3 linear equations in unknowns  $z^a$

- Matrix  $A$  depends on coupling constants and fields  $V_0^a$

$$A = \begin{pmatrix} 0 & -2\gamma_1 V_0^3 & \gamma_2 V_0^1 - \gamma_1 V_0^2 \\ 2\gamma_1 V_0^3 & 0 & \gamma_1 V_0^1 + \gamma_2 V_0^2 \\ -(\gamma_2 V_0^1 - \gamma_1 V_0^2) & -(\gamma_1 V_0^1 + \gamma_2 V_0^2) & 0 \end{pmatrix}$$

$$\gamma_1 = g_5 + g_7 \text{ and } \gamma_2 = g_4 + g_6 - 2g_2$$

- $\chi^a$  complicated functions of d.o.f.
- $\det(A)=0 \Rightarrow \chi^a$  cannot possibly be independent
- Solution exists for so-called **secondary constraint**

$$\phi_2 = \chi^1 (\gamma_1 V_0^1 + \gamma_2 V_0^2) + \chi^2 (\gamma_1 V_0^2 - \gamma_2 V_0^1) - \chi^3 2\gamma_1 V_0^3 \approx 0$$

**Assumption:** At least one of  $\gamma_1$  or  $\gamma_2$  does not vanish  $\Rightarrow$  solutions for Lagrange multipliers

$$z^1 = \frac{\chi^3 + \gamma_1 z^2 V_0^1 + \gamma_2 z^2 V_0^2}{\gamma_1 V_0^2 - \gamma_2 V_0^1}$$
$$z^3 = \frac{\chi^1 + 2\gamma_1 z^2 V_0^3}{\gamma_2 V_0^1 - \gamma_1 V_0^2}$$
$$z^2 \quad \text{solved from } \{\phi_2, H_1\} \approx 0$$

$\Rightarrow$  We are done!

**But**

Wrong number of constraints: 4 instead of 6

$$\phi_1^a \approx 0, \quad \phi_2 \approx 0$$

Consequence: Assumption

$$g_5 \neq -g_7$$
$$2g_2 \neq g_4 + g_6$$

leads to **inconsistent theory**

Thus  $\gamma_1 = \gamma_2 = 0$

–  $\Rightarrow$  None of the  $z^a$  can be solved at this stage (good)

–  $\Rightarrow$  **3** secondary constraints

$$\phi_2^a \equiv \{\phi_1^a, H_1\} \approx 0$$

– Secondary constraints conserved in time

$$\{\phi_2^a, H_1\} \approx 0$$

$\Rightarrow$  New system of 3 linear equations in  $z^a$

– Unique solution iff  $\det(\mathcal{M}) \neq 0$  for any field configuration

$$\mathcal{M}^{ab} \stackrel{\text{work}}{=} M_a^2 \delta^{ab} - \left( g^{bca} + g^{cba} \right) \partial_i V_i^c - \left( g^{ace} g^{bde} - 4h^{acbd} \right) V_i^c V_i^d - 4 \left( h^{abcd} + h^{acbd} + h^{adcb} \right) V_0^c V_0^d$$

**Strategy: Choose suitable field configurations**

**Example:**

$$V_i^a V_i^b = V_0^a = \partial_i V_i^1 = \partial_i V_i^2 = 0:$$

$$\det \mathcal{M} = \left( M^2 - 2 g_2 \partial_i V_i^3 \right)^2 \left( M_3^2 - 2 g_1 \partial_i V_i^3 \right)$$

can only be non-vanishing for arbitrary  $\partial_i V_i^3$ , if

$$g_1 = g_2 = 0$$

...

**Final result:**

**1. Correct number of constraints, i.e.,  $\gamma_1 = \gamma_2 = 0$ :**

$$g_7 = -g_5, \quad 2g_2 = g_4 + g_6 \quad (a)$$

**2. Analysis of determinant**

$$g_1 = g_2 = 0 \quad (b)$$

(a) and (b) imply

$$g^{abc} = -g^{bac}$$

Moreover

$$d_2 = -d_1,$$

$$d_1 \geq \frac{g_3^2}{2},$$

$$d_4 = -d_3,$$

$$d_3 \leq -g_4^2 - g_5^2,$$

$$d_5 = 0$$

–  $\Rightarrow$  **Consistent theory**, i.e., 6 constraints + Lagrange multipliers solved

- Summary after constraint analysis

Recall symmetry input: Lorentz invariance, parity, U(1) invariance

Symm. \ # of param.	before	tot.	after	tot.
U(1)	2 <i>M</i> , 7 <i>g</i> , 5 <i>h</i>	14	2 <i>M</i> , 3 <i>g</i> , 2 <i>h</i>	7
U(1) + <i>C</i>	2 <i>M</i> , 3 <i>g</i> , 5 <i>h</i>	10	2 <i>M</i> , 2 <i>g</i> , 2 <i>h</i>	6
SU(2)	1 <i>M</i> , 1 <i>g</i> , 2 <i>h</i>	4	1 <i>M</i> , 1 <i>g</i> , 2 <i>h</i>	4

+ two additional inequalities

We are not yet done!  $\Rightarrow$  Perturbative renormalizability

Constraints due to perturbative renormalizability **Without isospin symmetry:**

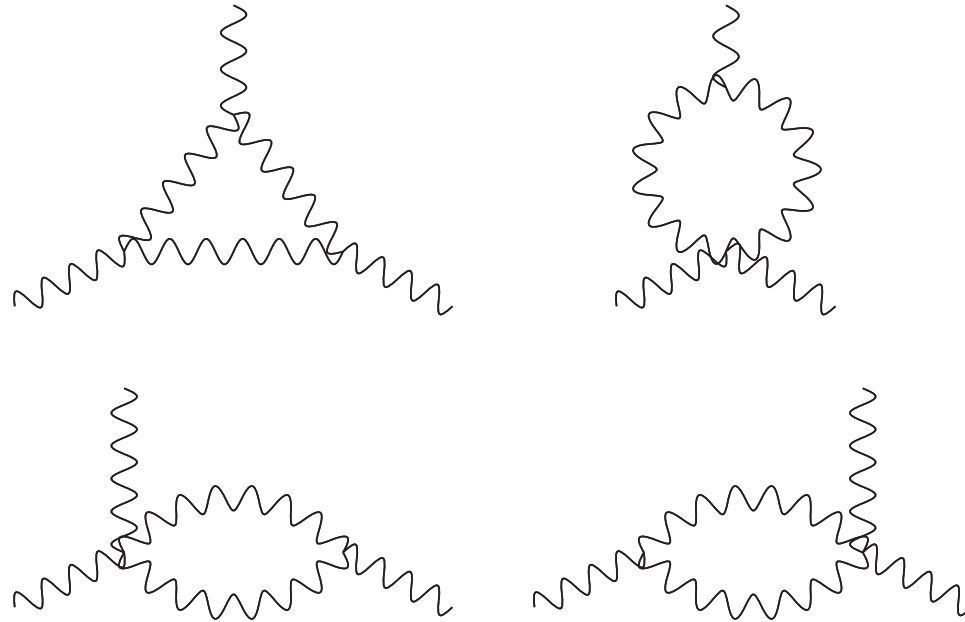
- Three coupling constants  $g_3, g_4, g_5$
- Two coupling constants  $d_1$  and  $d_3$
- Two masses  $M$  and  $M_3$

Require renormalizability in the sense of EFT:

Ultraviolet divergences of loop diagrams can be absorbed in the redefinition of masses, coupling constants, and fields of the most general effective Lagrangian containing **all** terms consistent with the assumed underlying symmetries.



- **Example: One-loop contributions to the three-vector vertex function**



**Logarithmically divergent parts of  $V^3V^3V^3$  and  $V^3V^3V^3V^3$  vertex functions + constraint analysis:  $\Rightarrow$**

$$g_4 \left( 5g_4^2 + 3g_5^2 + 3d_3 \right) = 0,$$

$$5g_4^4 + 2g_5^2g_4^2 + 5g_5^4 + 5d_3^2 + 2d_3 \left( g_4^2 + 5g_5^2 \right) = 0.$$

Unique solution reads

$$g_4 = 0, \quad d_3 = -g_5^2$$

- ...

- Final result

$$\begin{aligned} g^{abc} &= -g \epsilon^{abc}, \\ h^{abcd} &= \frac{1}{4} g^{abe} g^{cde}, \\ M_1 &= M_2 = M_3 = M \end{aligned}$$

- All couplings can be expressed in terms of one parameter  $g_3 = g$

- Perturbative renormalizability  $\Rightarrow$  SU(2) symmetry (including  $C$ )

## Interaction with fermions in isospin-symmetric limit (Universality I)

Interaction terms without derivatives only:

$$\mathcal{L}_F = \bar{\Psi} \left( i \partial_\mu \gamma^\mu - m + g_{VNN} \frac{\tau^a}{2} V_\mu^a \gamma^\mu \right) \Psi ,$$

Renormalization of **VVV** and **VVVV** vertex functions at one-loop level:

- Leading order in momentum expansion:  
Divergent parts are linear in momenta for **VVV** and have no momentum dependence for **VVVV**  
Counter terms  $\sim$

$$\begin{aligned} & -(\delta g + \frac{3}{2}g\delta Z_V), \\ & -\frac{1}{4}(2g\delta g + 2g^2\delta Z_V) \end{aligned}$$

- Calculate divergent parts of all one-loop contributions to self energies and vertex functions
- Perturbative renormalizability: UV divergences must cancel  $\Rightarrow$

$$g_{VNN} = 0, \quad \text{or} \quad g_{VNN} = g$$

- Universality

## Universality of the $\rho$ meson coupling in EFT (II) <sup>52</sup>

**Chirally invariant** effective Lagrangian including vector mesons

$$\mathcal{L} = \mathcal{L}_{\text{basic}} + \mathcal{L}_{\text{ct}} + \tilde{\mathcal{L}}_1$$

$\mathcal{L}_{\text{basic}}$  = free Lagrangians

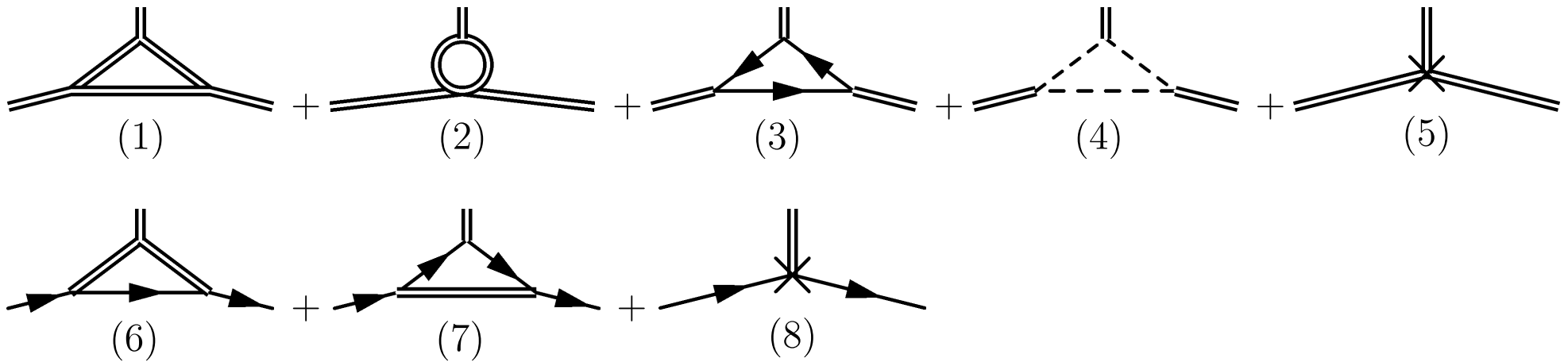
$$\begin{aligned} &+ \boxed{g_{\rho\pi\pi}} \varepsilon^{abc} \pi^a \partial_\mu \pi^b \rho^{c\mu} \\ &- \boxed{g} \varepsilon^{abc} \partial_\mu \rho_\nu^a \rho^{b\mu} \rho^{c\nu} \\ &- \frac{1}{4} \boxed{g^2} \varepsilon^{abc} \varepsilon^{ade} \rho_\mu^b \rho_\nu^c \rho^{d\mu} \rho^{e\nu} \\ &+ \boxed{g_{\rho NN}} \bar{\Psi} \gamma^\mu \frac{\tau^a}{2} \Psi \rho_\mu^a \end{aligned}$$

Universality (I)  $\Rightarrow$   $\boxed{g_{\rho NN} = g}$

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<sup>52</sup>D. Djukanovic, M. R. Schindler, J. Gegelia, G. Japaridze, S. Scherer, Phys. Rev. Lett. 93, 122002 (2004)

## Evaluate renormalized vertex diagrams



- $\rho\rho\rho$  vertex function  $\Rightarrow \boxed{\delta g} = F(g, g_{\rho\pi\pi})$

- $\rho\bar{\Psi}\Psi$  vertex function  $\Rightarrow \boxed{\delta g} = G(g, g_{\rho\pi\pi})$

$\Rightarrow$  consistency condition:  $F = G$

## Nontrivial solution

$$g_{\rho\pi\pi} = g$$

universality

Next step: Chiral symmetry <sup>53</sup>

$$g_{\rho\pi\pi} = \frac{M_\rho^2}{2gF^2}$$

Combine with universality  $\Rightarrow$

$$g^2 = \frac{M_\rho^2}{2F^2}$$

**KSRF relation**

54

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<sup>53</sup>S. Weinberg, Phys. Rev. 166, 1568 (1968);  
G. Ecker et al., Phys. Lett. B 223, 425 (1989)

<sup>54</sup>K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. 16, 255 (1966);  
Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966)

- Kawarabayashi & Suzuki

PCAC + current algebra  $\Rightarrow$

$$gg_{\rho\pi\pi} = \frac{M_\rho^2}{2F^2}$$

**NB: Chiral Ward identity**

$$\begin{aligned} T[\partial_\mu^x A_i^\mu(x) \partial_\nu^y A_j^\nu(y)] &= \partial_\mu^x \partial_\nu^y T[A_i^\mu(x) A_j^\nu(y)] \\ &+ \partial_\mu^x \delta^4(y-x) i \varepsilon_{ijk} V_k^\mu(y) \\ &+ \delta^4(x-y) i \hat{m} \delta_{ij} \bar{q}(x) q(x) \end{aligned}$$

universality as an **extra assumption**  $\Rightarrow$  KSRF relation



- Riazuddin & Fayyazuddin

PCAC + current algebra (but a different Ward identity similar to the Adler-Gilman relation of pion electroproduction, makes use of the analogue of the Goldberger-Treiman relation)  $\Rightarrow$

$$g_{\rho\pi\pi} = \frac{M_\rho^2}{2F^2}$$

**Dynamical assumption:**  $\pi^- \rightarrow \pi^0 + e^- + \bar{\nu}_e$  dominated by the  $\rho$  meson pole + CVC hypothesis  $\Rightarrow$  universality  $\Rightarrow$  KSRF relation

## Inclusion of the $\Delta(1232)$ into ChPT <sup>55</sup>

$$\Delta(1232) : \quad I(J^P) = \frac{3}{2} \binom{3^+}{2}$$

Description in terms of a vector-spinor isovector-isospinor

$$\Psi_{\mu,\alpha;i,m}$$

Too many components  $\Rightarrow$  Constraints

Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\frac{3}{2}} + \mathcal{L}_{\pi\Delta}$$

- Free-pion Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2$$

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<sup>55</sup>C. Hacker, N. Wies, J. Gegelia, S. Scherer, Phys. Rev. C 72, 055203 (2005);  
N. Wies, J. Gegelia, S. Scherer, Phys. Rev. D 73, 094012 (2006)

• Free- $\Delta$  Lagrangian <sup>56</sup>

$$\mathcal{L}_{\frac{3}{2}} = \bar{\psi}^\alpha \Lambda_{\alpha\beta}^A \psi^\beta,$$

where

$$\Lambda_{\alpha\beta}^A = -[(i\cancel{\partial} - m) g_{\alpha\beta} + iA (\gamma_\alpha \partial_\beta + \gamma_\beta \partial_\alpha) + \frac{i}{2}(3A^2 + 2A + 1)\gamma_\alpha \cancel{\partial} \gamma_\beta + m(3A^2 + 3A + 1) \gamma_\alpha \gamma_\beta]$$

with  $A \neq -1/2$  arbitrary real parameter

$$\begin{aligned} (i\cancel{\partial} - m_\Delta) \Psi^\mu &= 0, \\ \gamma_\mu \Psi^\mu &= 0, \\ \partial_\mu \Psi^\mu &= 0. \end{aligned}$$

$\mathcal{L}_{\frac{3}{2}}$  invariant under (often referred to as a point transformation)

$$\begin{aligned} \psi_\mu &\rightarrow \psi_\mu + a \gamma_\mu \gamma_\nu \psi^\nu, \\ A &\rightarrow \frac{A - 2}{1 + 4a}, \quad a \neq -\frac{1}{4} \end{aligned}$$

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<sup>56</sup> P. A. Moldauer and K. M. Case, Phys. Rev. 102, 279 (1956)

- Often used strategy: Construct interaction Lagrangian which is invariant under point transformation

We do **not impose** this constraint!

- $\pi\Delta$  interaction <sup>57</sup>

$$\mathcal{L}_{\pi\Delta} = -\bar{\Psi}^\mu \left[ \frac{g_1}{2} g_{\mu\nu} \gamma^\alpha \gamma_5 \partial_\alpha \phi \right. \\ \left. + \frac{g_2}{2} (\gamma_\mu \partial_\nu \phi + \partial_\mu \phi \gamma_\nu) \gamma_5 \right. \\ \left. + \frac{g_3}{2} \gamma_\mu \gamma^\alpha \gamma_5 \gamma_\nu \partial_\alpha \phi \right] \Psi^\nu$$

Apply Dirac's analysis using the Hamiltonian method:

Fields  $\psi^\mu$  and  $\psi^{\mu\dagger}$  + canonical momenta  $\pi_\psi^\mu$  and  $\pi_{\psi^\dagger}^\mu$   
 $\Rightarrow 2 \times 2 \times 4 \times 4 = 64$  components  $\Rightarrow 48$  constraints

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<sup>57</sup> T. R. Hemmert, B. R. Holstein, J. Kambor, J. Phys. G 24, 1831 (1998)

In a **consistent** theory

initial # of d.o.f – # of constraints = correct # of d.o.f.

⇒ Restrictions on the possible interaction terms

Analysis of constraints <sup>58</sup>

- 32 primary constraints
- time evolution ⇒ 8 secondary constraints
- time evolution ⇒ 8 secondary constraints
- all Lagrange multipliers have been solved and system terminates at 48 constraints iff

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<sup>58</sup>N. Wies, J. Gegelia, S. Scherer, Phys. Rev. D 73, 094012 (2006)

$$g_2 = Ag_1,$$

$$g_3 = -\frac{1}{2}(1 + 2A + 3A^2)g_1$$

- **NB: Demanding invariance under point transformations would generate**

$$g_2(A) = g_1 [2z_2 + (1 + 4z_2)A],$$

$$g_3(A) = 2g_1 \left[ z_3 + \left( \frac{1}{2} - z_2 + 4z_3 \right) A + \left( \frac{1}{4} + 4z_3 - 2z_2 \right) A^2 \right],$$

$z_2$  and  $z_3$  arbitrary

**Relations that follow from consistency are more stringent:**

$$z_2 = 0,$$

$$z_3 = -\frac{1}{4}$$

## Applications so far

- Mass of the nucleon <sup>59</sup>
- Pole of the  $\Delta$
- $\pi N$  scattering <sup>60</sup>
- Magnetic moment of the  $\Delta$  resonance <sup>61</sup>

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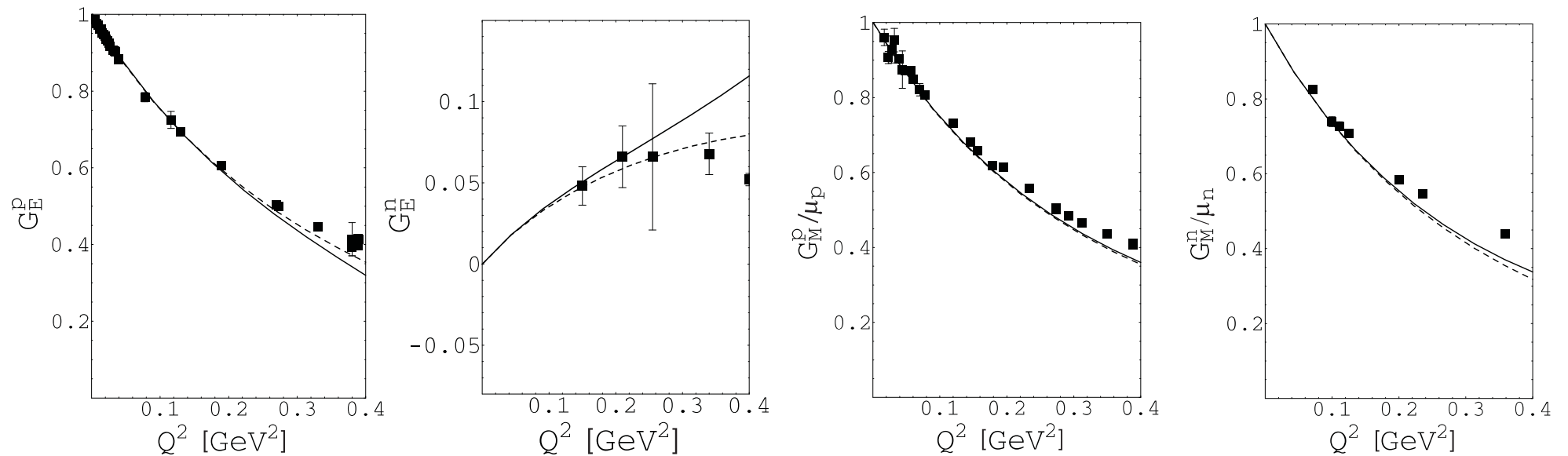
<sup>59</sup>C. Hacker, N. Wies, J. Gegelia, S. Scherer, Phys. Rev. C 72, 055203 (2005)

<sup>60</sup>N. Wies, thesis, Mainz, 2005

<sup>61</sup>C. Hacker, N. Wies, J. Gegelia, S. Scherer, Eur. Phys. J. A 28, 5 (2006)

## Electromagnetic form factors revisited II

E.m. form factors including vector mesons at  $\mathcal{O}(q^4)$  62



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<sup>62</sup>M. R. Schindler, J. Gegelia, S. Scherer, Eur. Phys. J. A 26, 1 (2005); data taken from J. Friedrich and Th. Walcher, Eur. Phys. J. A 17, 607 (2003)



## Consistent inclusion of vector mesons

Self-consistency in terms of constraints and perturbative renormalizability

→ Massive Yang-Mills Lagrangian <sup>63</sup>

$$\mathcal{L}_{\rho \text{ eff}} = -\frac{1}{2} \text{Tr} (\rho_{\mu\nu} \rho^{\mu\nu}) + M_{\rho}^2 \text{Tr}(\rho_{\mu} \rho^{\mu}),$$

$$\rho_{\mu} = \rho_{k\mu} \frac{\tau_k}{2}, \quad \rho_{\mu\nu} = \partial_{\mu} \rho_{\nu} - \partial_{\nu} \rho_{\mu} - ig [\rho_{\mu}, \rho_{\nu}]$$

Two parameters

1.  $\rho$ -meson mass  $M_{\rho}$  in the chiral limit
2. Coupling strength  $g$

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<sup>63</sup>D. Djukanovic, J. Gegelia, S. Scherer, Int. J. Mod. Phys. A 25, 3603 (2010)

## Interaction with pions

Choose  $\rho$  mesons to transform **inhomogeneously** under  $(V_L, V_R) \in \mathbf{SU}(2)_L \times \mathbf{SU}(2)_R$

$$\rho_\mu \mapsto K \rho_\mu K^\dagger - \frac{i}{g} \partial_\mu K K^\dagger,$$

$$K(V_L, V_R, U) = \sqrt{V_R U V_L^\dagger}^{-1} V_R \sqrt{U}.$$

Mass term remains chirally invariant through the replacement

$$\rho_\mu \rightarrow \rho_\mu - (i/g)\Gamma_\mu$$

## Effective chiral Lagrangian

$$\begin{aligned} \mathcal{L}_{\pi\rho} = & -\frac{1}{2} \text{Tr} (\rho_{\mu\nu} \rho^{\mu\nu}) + M_\rho^2 \text{Tr} \left[ \left( \rho_\mu - \frac{i}{g} \Gamma_\mu \right) \left( \rho^\mu - \frac{i}{g} \Gamma^\mu \right) \right] \\ & + \frac{1}{2} d_x \text{Tr} (\rho_{\mu\nu} f_+^{\mu\nu}) + \dots \end{aligned}$$

$$\mathcal{L}_{\pi\omega}^{(3)} = -f_\omega (\partial^\mu \omega^\nu - \partial^\nu \omega^\mu) v_{\mu\nu}^{(s)} + \dots$$

## $\pi$ VN interaction

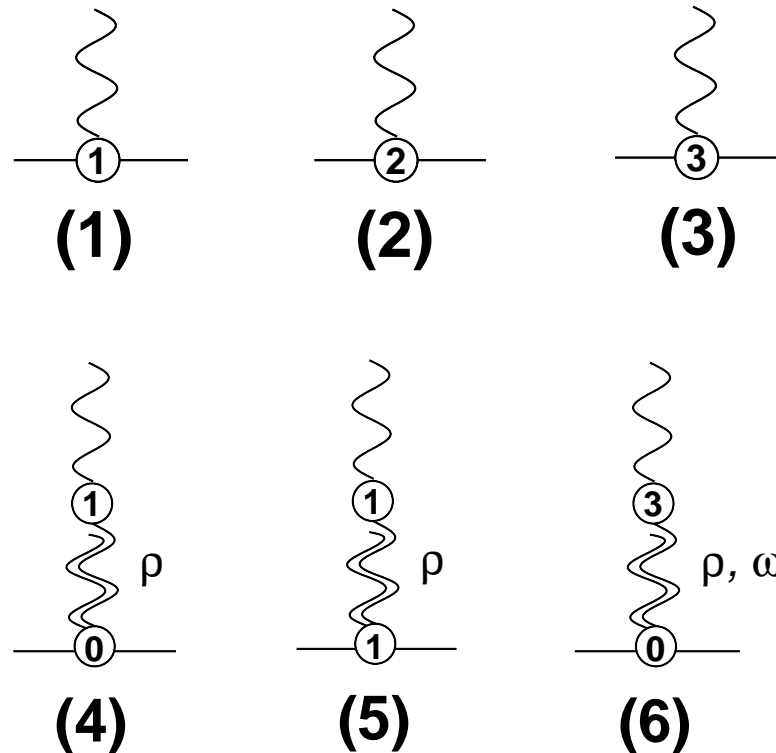
$$\mathcal{L}_{\pi VN} = \bar{\Psi} \left[ g \left( \rho_\mu - \frac{i}{g} \Gamma_\mu \right) + \frac{1}{2} g_\omega \omega_\mu \right] \gamma^\mu \Psi + \frac{G_\rho}{2} \bar{\Psi} \rho_{\mu\nu} \sigma^{\mu\nu} \Psi + \dots$$

## Perturbative renormalizability:

- **Universality of  $\rho$ -meson coupling:**  $g_{\rho\pi\pi} = g_{\rho NN} = g$
- **KSRF-relation:**  $M^2 = 2g^2 F^2$

# Feynman diagrams including vector mesons up to and including $\mathcal{O}(q^3)$ <sup>64</sup>

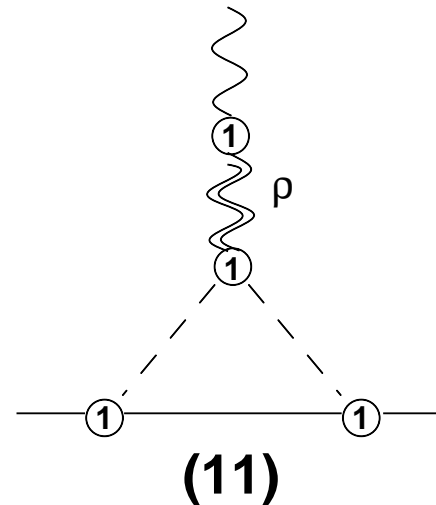
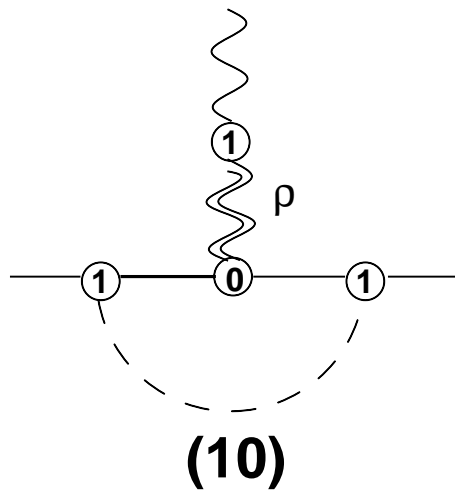
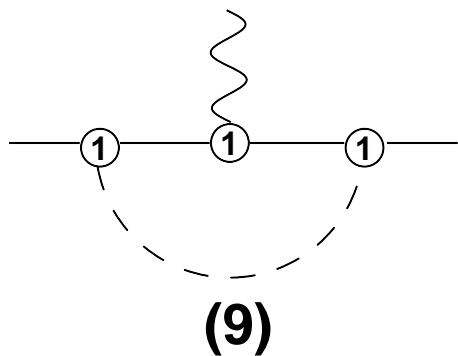
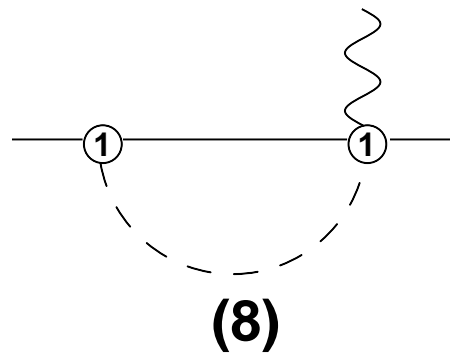
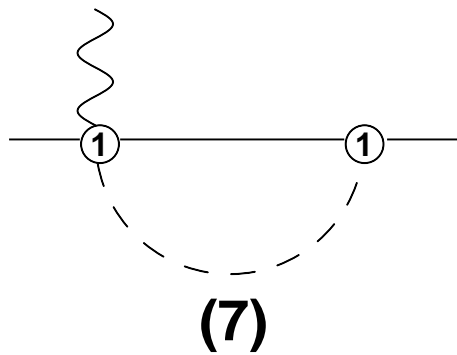
## Tree-level diagrams



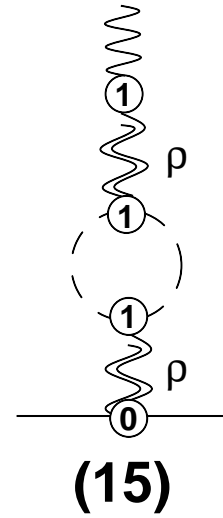
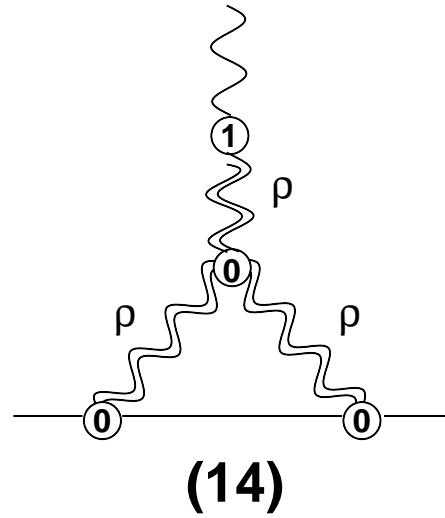
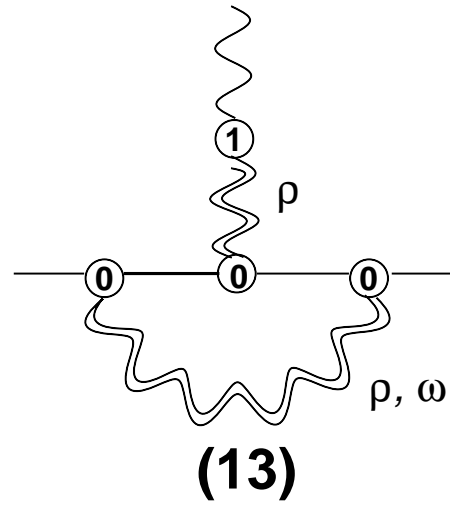
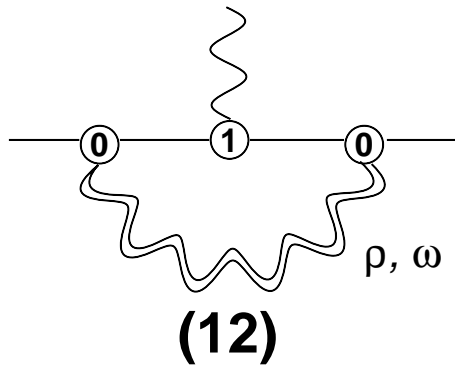
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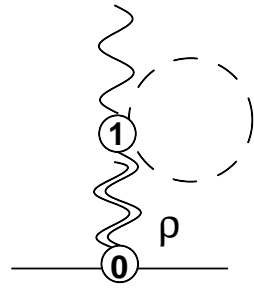
<sup>64</sup>T. Bauer, J. C. Bernauer, S. Scherer, Phys. Rev. C 86, 065206 (2012)

# One-loop diagrams (set 1)

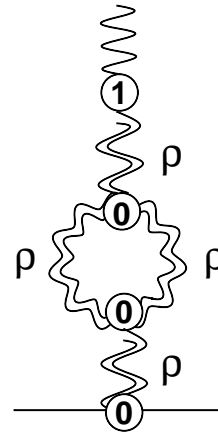


## One-loop diagrams (set 2)

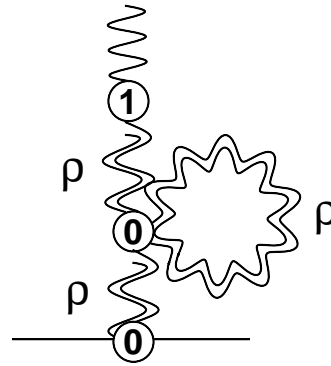




**(16)**

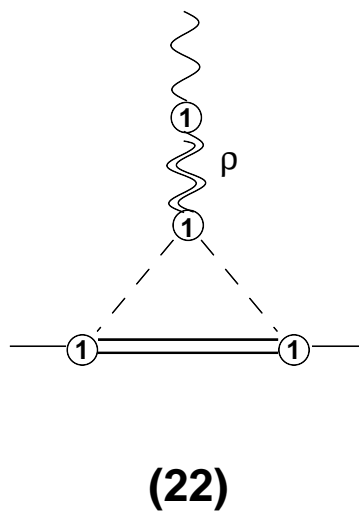
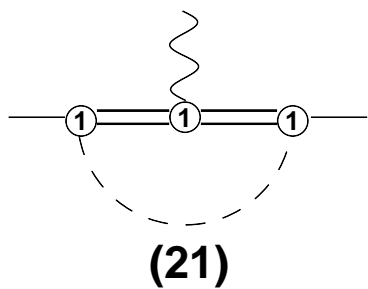
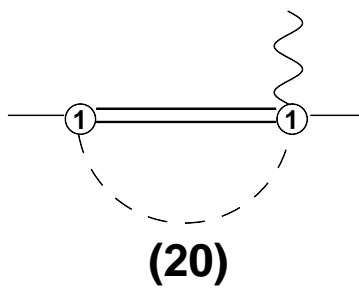
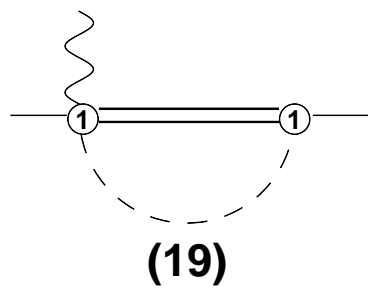


**(17)**



**(18)**

## One-loop diagrams (set 3)





## Input parameters (LECs) from other sources

### Masses

$$m_N = 938 \text{ MeV},$$

$$M_\pi = 140 \text{ MeV},$$

$$M_\rho = 775 \text{ MeV},$$

$$M_\omega = 783 \text{ MeV},$$

$$m_\Delta = 1210 \text{ MeV}$$

### Decay constant and couplings

$$F_\pi = 92.2 \text{ MeV},$$

$$g_A = 1.27,$$

$$g = 5.93,$$

$$g = 1.13.$$

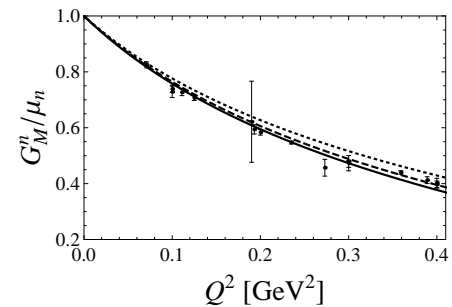
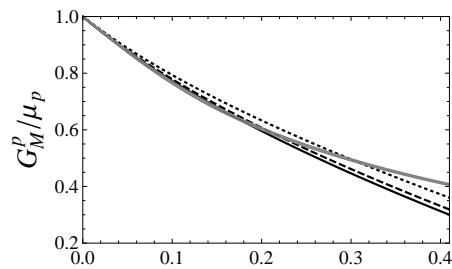
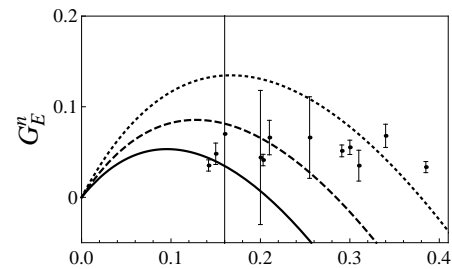
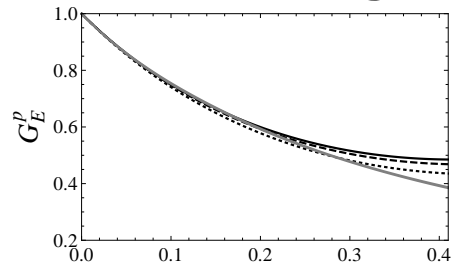
Fit parameters ( $\Delta$ ,  $\rho$ ,  $\omega$ )

$Q_{\max}^2$	$g_\omega$	$f_\omega \cdot g_\omega$	$G_\rho$	$d_6$	$d_7$	$d_x$	$\chi_{\text{red}}^2$
<b>0.2</b>	-1.06	-0.106	-4.84	<b>1.67</b>	-0.282	-0.506	<b>1.50</b>
<b>0.3</b>	-1.27	-0.127	-4.05	<b>1.55</b>	-0.233	-0.512	<b>4.21</b>
<b>0.4</b>	-1.92	-0.192	-1.98	<b>1.50</b>	-0.211	-0.547	<b>25.30</b>

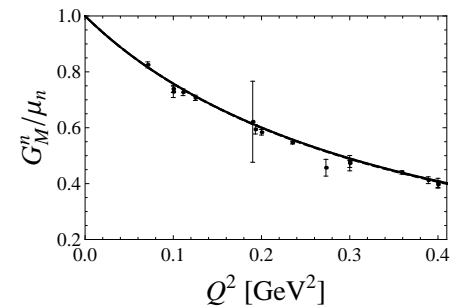
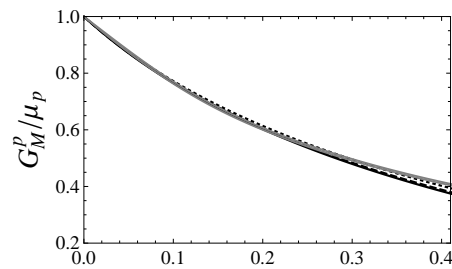
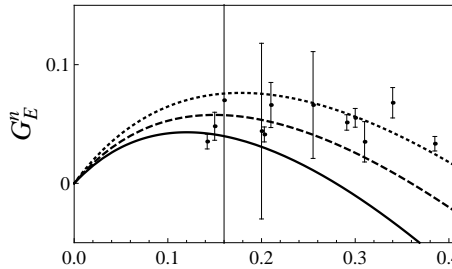
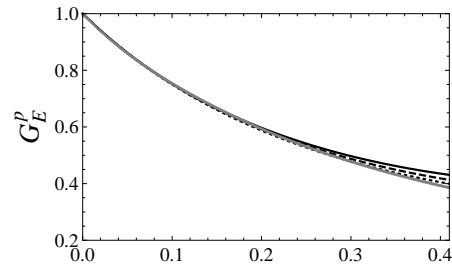
Fit parameters without  $\Delta$  ( $\rho$ ,  $\omega$ )

$Q_{\max}^2$	$g_\omega$	$f_\omega \cdot g_\omega$	$G_\rho$	$d_6$	$d_7$	$d_x$	$\chi_{\text{red}}^2$
<b>0.2</b>	<b>5.13</b>	<b>0.513</b>	-16.90	<b>0.629</b>	<b>0.0909</b>	-0.134	<b>1.45</b>
<b>0.3</b>	<b>4.91</b>	<b>0.491</b>	-17.13	<b>0.507</b>	<b>0.0991</b>	-0.118	<b>1.74</b>
<b>0.4</b>	<b>4.49</b>	<b>0.449</b>	-16.58	<b>0.490</b>	<b>0.0934</b>	-0.130	<b>3.57</b>

## Fits including $\Delta$



## Fits without $\Delta$



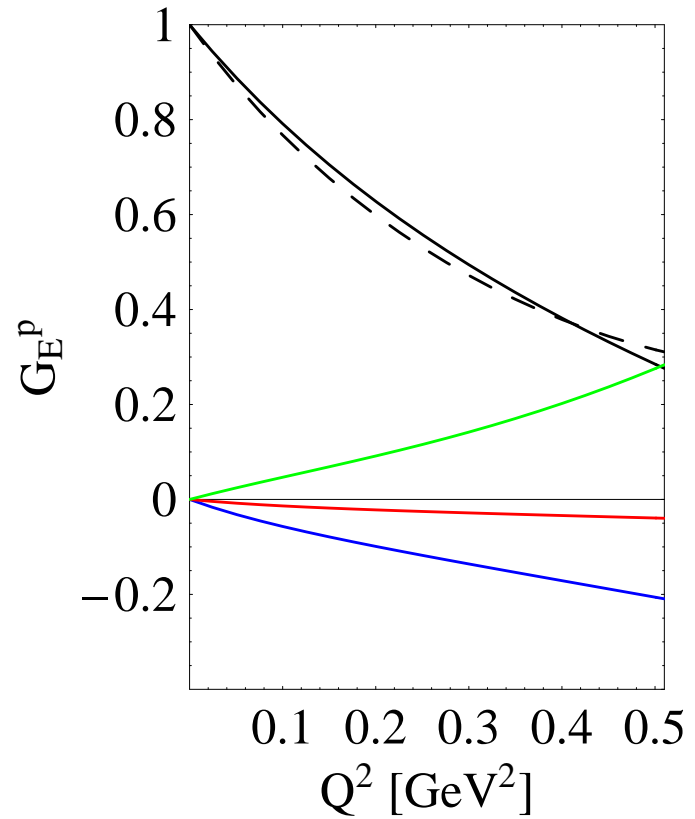
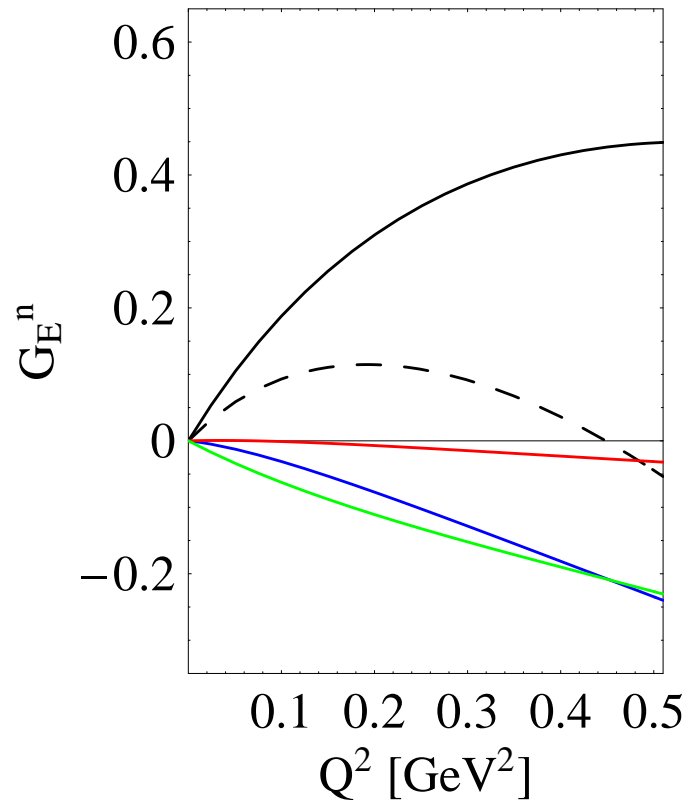
**Solid lines:**  
 $Q_{\max}^2 = 0.2 \text{ GeV}^2$

**dashed lines:**  
 $Q_{\max}^2 = 0.3 \text{ GeV}^2$

**dotted lines:**  
 $Q_{\max}^2 = 0.4 \text{ GeV}^2$

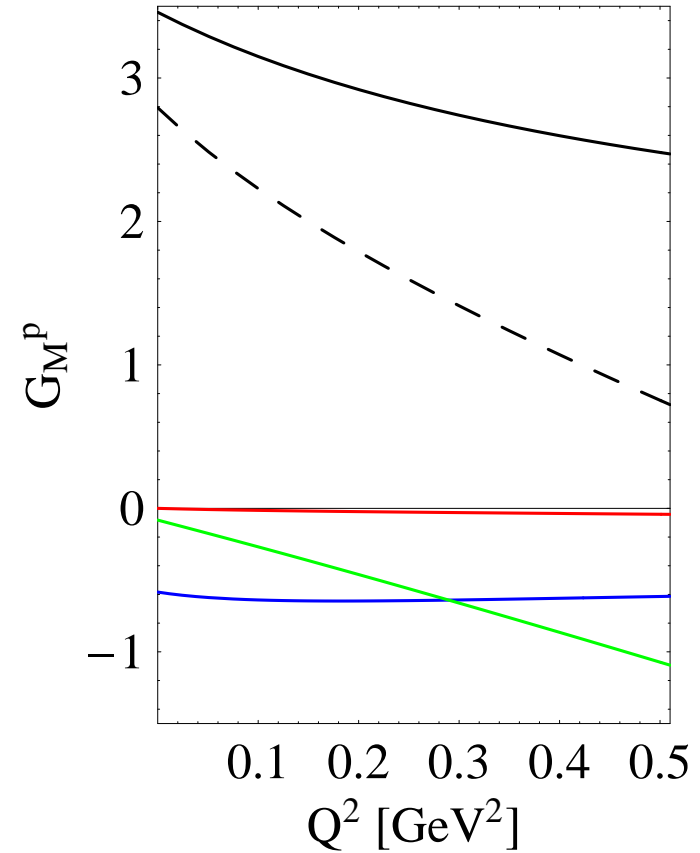
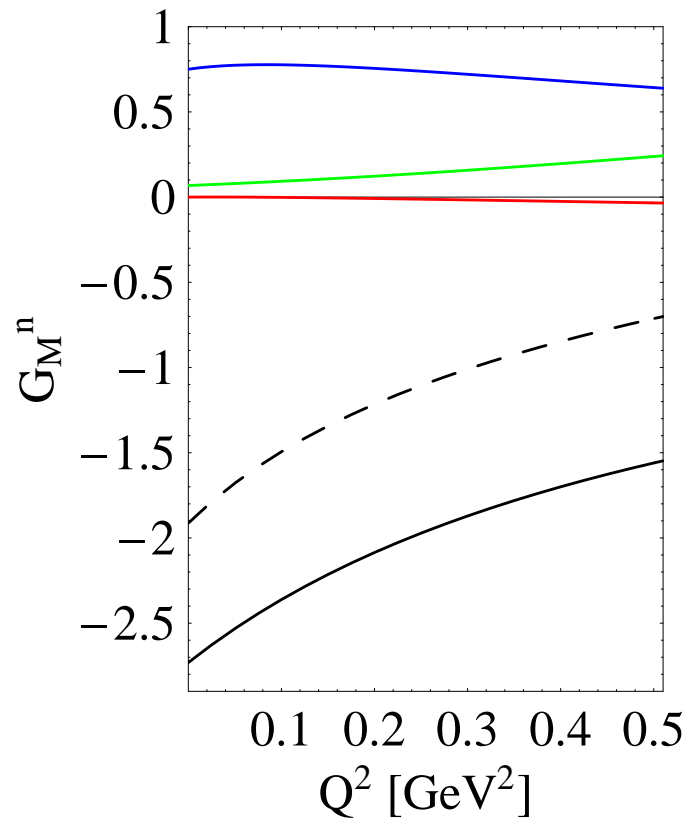
**grey bands:**  
empirical fits of the  
form factors to the  
measured cross  
sections

## Decomposition into individual contributions



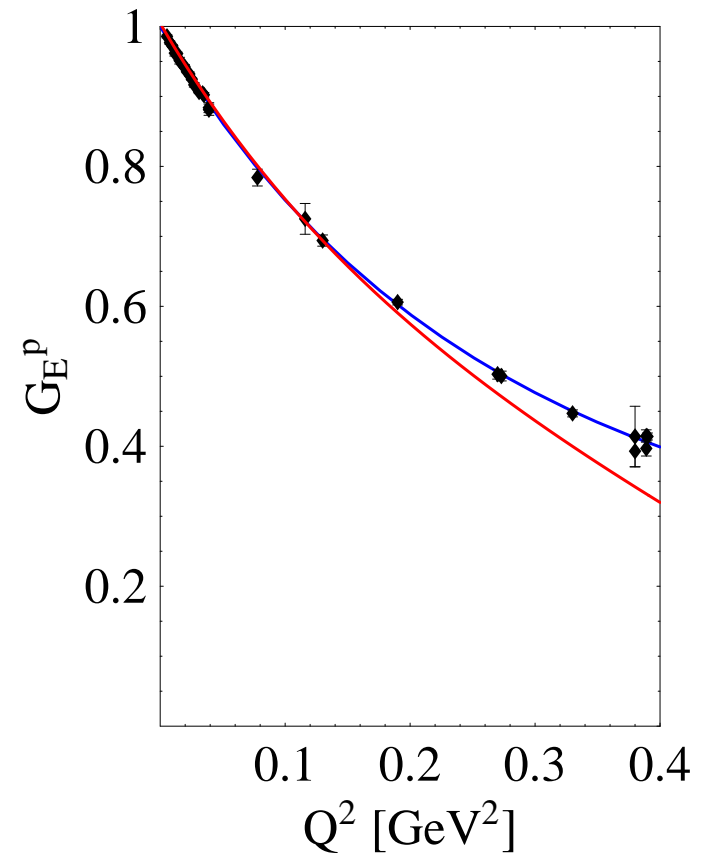
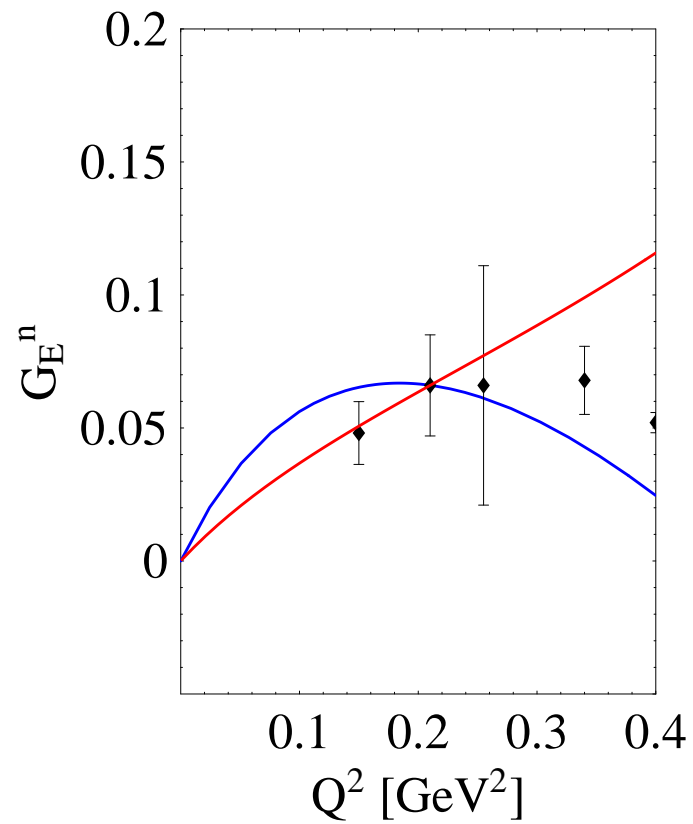
**Full lines: tree-level results; dashed lines: full contribution; blue lines: contribution of set 1; red lines: contribution of set 2; green lines: contribution of set 3**

## Decomposition into individual contributions



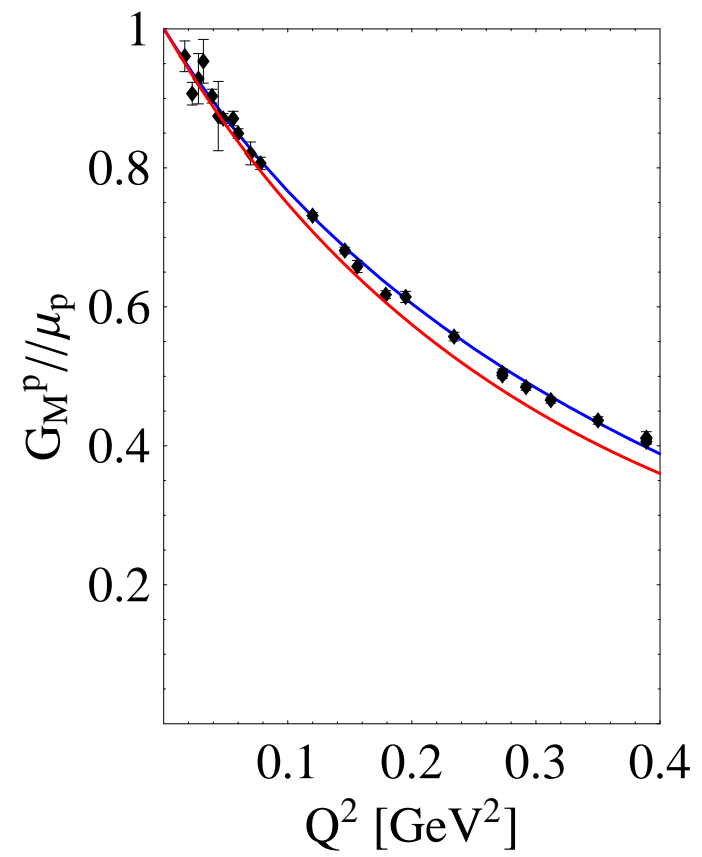
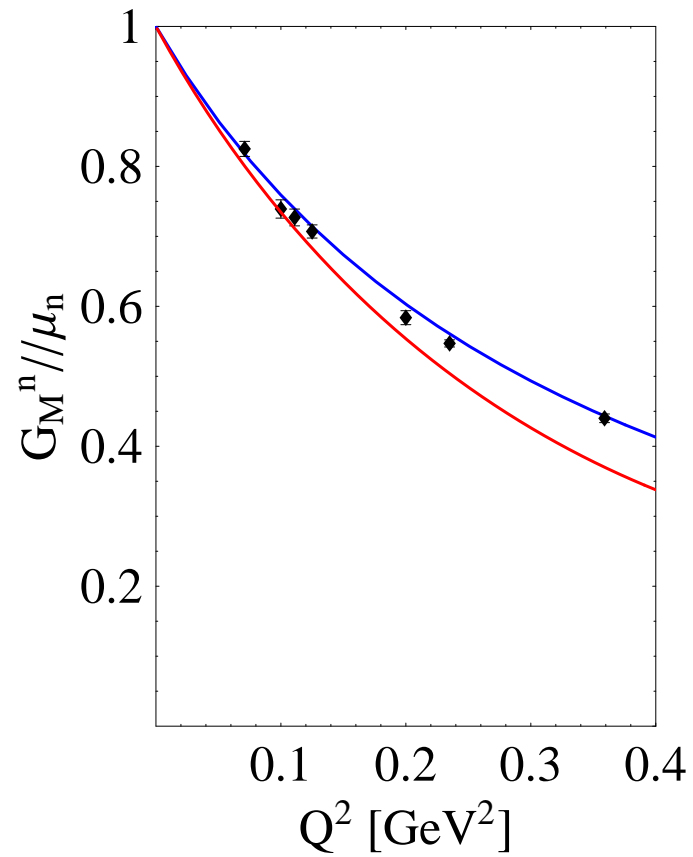
Full lines: tree-level results; dashed lines: full contribution; blue lines: contribution of set 1; red lines: contribution of set 2; green lines: contribution of set 3

## Comparison new vs. old vector-meson calculation



Red lines: old calculation; blue line: new calculation

## Comparison new vs. old vector-meson calculation



Red lines: old calculation; blue line: new calculation