Phenomenology of Nucleon Form Factors

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The most simple reaction that can be studied theoretically and experimentally is the process involving electrons and protons. In spite of its elementarity, electron-proton elastic scattering and the crossed channels as nucleon-antinucleon to (or created by) an electron positron pair are actively studied since decades. Assuming that the colliding particles interact by exchange of one virtual photon, the transferred momentum squared (the mass of the virtual photon) probes the dynamical structure of the nucleon at the corresponding internal scale.

The differential cross section and the polarization observables in these reactions are expressed in terms of form factors, which contain unique information on the nucleon structure: form factors parametrize the internal structure of a composite particle and describe its dynamical properties. The knowledge of form factors constitutes a stringent test for any model which, after the static properties of a particle, like masses or magnetic moments, should be able to reproduce charge and magnetic distributions.

The experimental achievements: high energy accelerators, high intensity beams, high resolution spectrometers, polarized targets, hadron polarimeters, full coverage detectors.. open the possibility of very precise measurements in an unexplored kinematical region. A wide program is ongoing or is planned at facilities in the GeV range: electron accelerators, such as Jefferson Lab (Newport News), electron-positron colliders such as VEPPIII (Novosibirk), BEPCII (Beijing), proton-antiproton colliders such as FermiLab and the future FAIR facility, at Darmstadt.

From the theoretical point of view, the precise knowledge of the form factors in a wide kinematical range gives the best insight in the transition region, between the non perturbative domain where the nucleon is best described by constituent quarks and meson cloud, and the perturbative region where QCD can be applied and the nucleon appears as a confined system of quarks and gluons. Analytical and model independent properties of form factors are a guide for modelization of the nucleon structure.

After an historical and pedagogical introduction into this field, a formal derivation of electromagnetic form factors for the scattering and the annihilation channels, as well as discussion of the recent data, and of new ideas in the understanding of the reaction mechanism, is given.

1. Introduction

The experimental determination of the elastic proton electromagnetic form factors (FFs) at large momentum transfer is presently of large interest, due to experimental developments which open the possibility to achieve new kinematical regions and very high precision. In particular, polarization experiments have been made possible by polarized electron beams at high intensity and proton polarimetry in the GeV energy region, as suggested many years ago.[1–3]

Hadron FFs are considered fundamental quantities, as they characterize the internal structure of a non pointlike particle. They contain dynamical information on the electric and magnetic currents of hadrons, and are experimentally accessible through differential cross section and polarization observables. Theoretically FFs enter in the expression of the electromagnetic current. Any hadron theory, that reproduces the static properties such as masses and magnetic moments, should be able to describe also the dynamics of the charge and magnetic distributions, i.e., the electromagnetic FFs.

In a P and T invariant theory, the structure of any particle of spin S is parametrized in terms of $(2S + 1)$ FFs. Protons and neutrons are described by two FFs, electric G_E and magnetic G_M, which are functions of one kinematical variable, physically representing the internal scale. The deuteron (spin one particle) is described by three form factors, charge, electric, and quadrupole. The α particle, spin zero, has one form factor.

The normalization of these FFs is related to the charge and the magnetic moment of the hadron and corresponds to the static value which can be observed through low energy electron elastic scattering on hadrons, at the photon point. Schematically, one can say that at small momenta (large internal distances) FFs probe the size of the nucleus. At high energies (short distances) they probe the quark and gluon structure. Their behavior should follow scaling laws, predicted by perturbative quantum-chromodynamics (pQCD). In this respect, the precise knowledge of FFs in a wide kinematical region should probe the transition region, from non perturbative to perturbative QCD.

The traditional way to measure proton electromagnetic FFs consists in the measurement of electron-proton elastic scattering, assuming that the interaction occurs through the exchange of a virtual photon, of four momentum squared $Q^2 = -q^2$. The differential cross section at a fixed value of Q^2 is depends linearly on $\cot^2(\theta/2)$ (where θ is the electron scattering angle). The slope and the intercept allow to determine G_E and G_M. This is a specific characteristic of the one photon exchange mechanism. This method was proposed first by N. M. Rosenbluth.[4]

Polarization phenomena play a major role (except for spin zero particles), as they contain unique information on the imaginary part of amplitudes (amplitudes are, in general, complex functions). Being related to interference of amplitudes, they are very sensitive to small contributions. Elastic electron hadron scattering has been the privileged reaction to access FFs. Assuming one photon exchange, a simple and elegant formalism, which will be illustrated in these lectures, relates all observables, cross section and polarization phenomena, to hadron FFs.

The idea that double spin polarization observables in elastic electron proton (ep) scattering (with longitudinally polarized electrons on a polarized target, or on an unpolarized target, measuring the transverse polarization of the scattered proton) carry the information on the product $G_E G_M$ was firstly suggested by A. I. Akhiezer and M. P. Rekalo[1] but was only recently applied. Besides the expected large precision achieved, the surprising fact, was that the data revealed a Q^2-dependence of the ratio $\mathcal{R} = \mu G_E/G_M$ (μ is the proton magnetic moment) which deviates from unity, as was previously commonly assumed.

In case of the neutron, the measurements are even more difficult, as the electric FF is small (the static value is zero). As there is no free neutron target, one has to use either a deuteron or an ^3He target, and then correct for nuclear effects. In the neutron case, too, the polarization method allows to extend the measurements in the scattering region at larger Q^2 values with higher precision.

Inconsistencies appeared among the results from polarized and unpolarized experiments. The ratio $\mu G_E/G_M$ measured from the ratio P_ℓ/P_t (the longitudinal and transverse polarization of the recoil proton in ep scattering induced by longitudinally polarized electrons) shows a monotone decreasing with Q^2, whereas the individual determination of G_E and G_M from the Rosenbluth separation suggests a constant behavior. No shortcoming has been found neither in the experiments or in the data analysis, which are based on the same theoretical background (the lowest order diagrams for ep elastic scattering). Therefore, the attention has been focused to higher order corrections in the power of α, radiative corrections in α^n including the interference between one and two photon exchange. This puzzle has given rise to many speculations and different interpretations, suggesting further experiments (for a review, see[5]).

Applying crossing symmetry considerations, the same physical information can be extracted from the annihilation reactions: $\bar{p} + p \leftrightarrow e^+ + e^-$ through the measurement of a precise angular distribution. However, the kinematical variables scan a different region, called the time-like (TL) region, because the momentum transfer squared is positive here (i.e, the time component of the four momentum transfer squared, q^2, is larger than the space component). The region accessible through the scattering channel is therefore denoted as space-like (SL) region.

FFs are assumed to be analytical functions of q^2.[6] In the general case, reaction amplitudes are complex functions of the relevant kinematical variables. Analyticity and unitarity constrain FFs to be real in SL region, and complex in TL region. Up to now, no individual determination of FFs has been done in TL region, due to the low statistics. FFs have been determined under the assumption $G_E = G_M$ or G_E=0.[7] Attempts of measuring the FF ratio were done by PS170[8] and BABAR[9] collaborations.

The possibility of better measurements has inspired experimental programs to measure hadron form factors at JLab, Frascati and at future machines, such as FAIR, both in SL and in TL regions. Electron beams in the GeV range are available at MAMI and JLab, with high intensity and high polarization, large acceptance spectrometers, hadron polarized targets, and hadron polarimeters. In colliding mode, the VEPP2 facility at Novosibirsk and the BES facility at BEPC provide 4π detection with high luminosity e^+e^- collisions. High intensity hadron and particularly antiproton beams will be available at PANDA (FAIR) in near future.

From a theoretical point of view, the new results obtained with the polarization method have stimulated a revision of the nucleon models. The interpretation of FFs as the Fourier transforms of charge and magnetization densities is exact only in non relativistic approximation or in the Breit frame, where the four components of the momentum can be reduced to three. Recent model dependent pictures of the proton structure have been derived. In particular, form factors are specific integrals of generalized parton distributions, and they constitute, in this respect, an experimental constraint for theses functions. Different classes of models have been developed in the non perturbative region: soliton models, constituent quarks, di-quark models, vector meson dominance, dispersion relations ... (for a review, see[5]). However, not all of them are able to describe the existing data on the four nucleon FFs (electric, magnetic, neutron and proton) and not all of them contain the necessary analytical properties to describe both the SL and TL regions.[10]

2. History

In 1961 R. Hofstadter got the Nobel prize, "for his pioneering studies of electron scattering in atomic nuclei and for his thereby achieved discoveries concerning the structure of the nucleons". In his Nobel lecture one can read *"Over a period of time lasting* *at least two thousand years, Man has puzzled over and sought an understanding of the composition of matter. It is no wonder that his interest has been aroused in this deep question because all objects he experiences, including, even his own body, are in a most basic sense special configurations of matter. The history of physics shows that whenever experimental techniques advance to an extent that matter, as then known, can be analyzed by reliable and proved methods into its "elemental" parts, newer and more powerful studies subsequently show that the "elementary particles" have a structure themselves. Indeed this structure may be quite complex, so that the elegant idea of elementarity must be abandoned."*

The first experimental evidence for a composite structure of the proton, arising from charge and magnetization currents, dynamically changing with the distance (probed by the virtual photon in ep elastic scattering), was given in a series of experiments at the Stanford accelerator SLAC, based on the Rosenbluth separation.[11]

In this chapter we recall the milestones of our present knowledge on FFs.

- Rutherford scattering
  - 1909: Experiments of Geiger and Marsden. The cross section for the scattering of electrons in the Coulomb field of a nucleus of charge Z, is given by the Rutherford formula (1911).[12] It applies to non relativistic, spin zero, pointlike particle scattering. It was used to measure the 'size' of the target and to introduce the concept of atomic nucleus[*].
  - 1968: DIS "deep inelastic scattering" experiments in which very energetic electrons were scattered off protons showed that all the mass and charge of the proton is concentrated in smaller components, then called "partons". Partons were later identified with quarks (Friedman, Kendall and Taylor, Nobel Prize 1991).
  - 1967: First order extension of the Rutherford formula, valid at high energy.[13]
  - 1975-79: Extension to higher orders (eikonal approximation).[14,15]
  - >1980: Extension to heavy ions/ polarization observables.[16,17]
- Beyond the Rutherford formula
  - 1929: N.F. Mott derives a formula for relativistic nuclei, that holds for scattering of spin 1/2 pointlike particles.[18]
  - 1950: M.N. Rosenbluth extends the formalism to composite targets.[4]
  - 1961: R. Hofstadter receives the Nobel Prize, for experiments at SLAC, on unpolarized ep scattering, at fixed Q^2, doing the first experimental determination of G_E and G_M.[11]
  - 1958-1967: Polarization in ep scattering (Kharkov school,[19] and[20]). A.I Akhiezer and M.P. Rekalo give the explicit derivation of polarization observables for elastic ep scattering in terms of form factors.[1,2]
  - later, after 1997: Polarization experiment at MIT, JLab[21] and Refs. therein.
    - *better precision* (large sensitivity to the small G_E contribution)
    - *determination of the sign of FFs*.
- Time-like region
  - 1962: Cross section and single polarization in terms of FFs in the annihilation process $p + \bar{p} \to e^+ + e^-$ (A. Zichichi, S. M. Berman, N. Cabibbo, R Gatto[22]).
  - 1983-1994: First TL measurements with antiprotons at LEAR(CERN): PS170.[8]
  - 1998: First TL measurements at FENICE (Frascati), with e^+e^- collisions for proton and neutron FFs.[23]
  - 1997-2003: E760,[24] E835[7] with antiprotons at FermiLab.
  - 2002: Threshold measurements at BES.[25]
  - 2005: ISR in BABAR e^+e^- colliders.[9]
  - after 2010: Experiments at BESIII.
- Reaction mechanism
  - 1970-73: Experimental and theoretical studies of two photon exchange.[26–30]
  - 1999-2006: Model independent properties of unpolarized and polarized scattering[31,32] and annihilation[33] in presence of two photon exchange.
  - 2006-today: Model calculations including proton structure[34] and revision of nucleon models.
- Radiative corrections:
  - 1949: J.S. Schwinger calculates photon emission in pure QED scattering[35]
  - 1969: L.W. Mo and Y.S. Tsai calculate at first order the radiative corrections for electron hadron scattering.[36]

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[*]Different sites have been built to play with.
For example, see http://waowen.screaming.net/revision/nuclear/rssim.htm

- 1985: E.A. Kuraev and V.S. Fadin include higher orders using the electron structure function method, and apply those to elastic and deep inelastic scattering.[37]
- 2000: L.C. Maximon and J.A. Tjon revise the work of Ref.[36] on ep scattering including (partly) the structure of the proton.[38]
- $>$ 2000: Radiative corrections to polarization phenomena in ep elastic scattering.[39–41]

3. Basic concepts

As a first exercise, we consider here the elastic scattering of *structureless* particles,

$$a(p_a) + b(p_b) \to c(p_c) + d(p_d), \tag{1}$$

(the four momenta are indicated in parenthesis) which interact through the Coulomb potential $H_1 = U(\vec{r})$. The Coulomb potential between the target and the projectile $U(r)$ is spherically symmetric, directly proportional to the charges and inversely to the distance:

$$U(r) = \frac{Z_a Z_b e^2}{r}. \tag{2}$$

In order to take into account the screening effects of the electrons surrounding the atomic nucleus (and also to avoid divergences), a damping function is added and the Coulomb potential is usually as:

$$U(r) = \frac{Z_a Z_b e^2}{r} e^{-r/\lambda}, \tag{3}$$

where $\lambda \sim 10^{-8}$ cm $\sim 10^5$ fm is of the order of the dimensions of the atom.

3.1. *Reminder on perturbation theory*

The elements of the scattering matrix, S_{fi} are the probability amplitudes for the reaction $i \to f$. The initial state of the system, $|i>$, after an interaction can be written as a superposition of possible final free particle states $|f>$:

$$|\Psi_i>= \sum_f |f><f|S|i>= \sum_f |f> S_{fi} \tag{4}$$

where $|S_{fi}|^2$ is the probability of the transition $i \to f$. $S \equiv U(-\infty, \infty)$, $U(t, t_0)$ is the time evolution operator.

The scattering amplitude T is defined as:

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i) T_{fi}, \tag{5}$$

where δ_{fi} is the Kronecker symbol, which gives the superposition $< f|i >$ in the absence of interaction. The Dirac function

$$\delta^4(p_f - p_i) = \delta^3(\vec{p}_f - \vec{p}_i)\delta(E_f - E_i), \tag{6}$$

insures that each component of the four vector energy-momentum has to be conserved.

In a scattering process, the matrix element can be expressed using the perturbation theory. The Hamiltonian which describes the evolution of the system can be decomposed as:

$$H = H_0 + H_1, \tag{7}$$

where H_0 is the free particle Hamiltonian and H_1 is the interaction Hamiltonian. The time evolution is given in the Heisenberg representation by

$$H'_I(t) = e^{iH_0(t-t_0)} H' e^{-iH_0(t-t_0)}. \tag{8}$$

Assuming that $H'(t)$ can be treated as a perturbation, it is convenient to develop the S matrix in a series of terms which contain the product of operators H'_I:

$$S = U(-\infty, \infty) = \sum_{n=1}^{\infty} S^n = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\infty}^{\infty} dt_n..dt_1 T[H'_I(t_n)...[H'_I(t_1)] \tag{9}$$

where T is the time ordering operator, particularly important when $H'_I(t)$ and $H'_I(t')$ do not commute.

The first terms of the development (9) are:

$$S^1 = -i\int_{\infty}^{\infty} H_I'(t_1)dt_1,$$
$$S^2 = -\frac{1}{2}\int_{\infty}^{\infty} dt_1 \int_{\infty}^{\infty} dt_2 T\left[H_I'(t_2)H_I'(t_1)\right]. \tag{10}$$

There is of course a one to one correspondence with the matrix T: a corresponding term of the same order and corresponding elements $T_{fi} = \sum_{n=0}^{\infty} T_{fi}^n$. **The Born approximation consists in keeping only the term** $n = 1$. For our interest here, it is applied to processes which involve electromagnetic and weak interactions.

3.2. *Derivation of the Rutherford formula: analogy with optics*

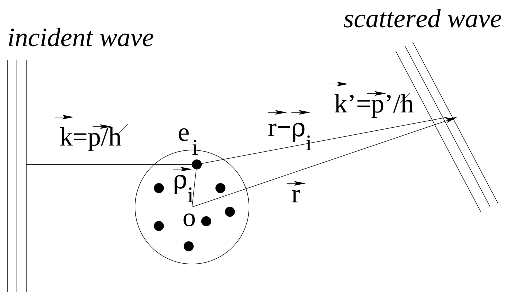

Fig. 1. Schematic view of elastic scattering on a composite object.

In quantum mechanics, the particle-wave duality requires that a particle of three momentum $\vec{p}$ is associated to a plane wave vector $\vec{k} = \vec{p}/\hbar$. If a plane wave scatters off a charge e_i at a position ρ_i, it generates a spherical wave, that can be observed at large distances as a plane wave $\vec{k}' = \vec{p}\,'/\hbar$. The amplitude of the scattered wave in the point defined by $\vec{r}$ is:

$$A_i = f e_i e^{i\vec{k}\cdot\vec{\rho}_i} e^{i\vec{k}'\cdot(\vec{r}-\vec{\rho}_i)} = f e^{i\vec{k}'\cdot\vec{r}} e_i e^{iq\cdot\vec{\rho}_i} \tag{11}$$

where f is the amplitude on the unit charge, $f = Z_a e$, which is the same for all constituent particles, $\vec{r} - \vec{\rho}_i$ is the vector from the observation point to the charge i, and $\vec{q} = \vec{p}_i - \vec{p}_f$ is the momentum transfer. The factor $e^{i\vec{k}\cdot\vec{\rho}_i}$ defines the phase of the incident plane wave at the interaction point, and $e^{i\vec{k}'\cdot(\vec{r}-\vec{\rho}_i)}$ determines the phase of the scattered wave at the observation point. Similarly to optics, the total scattered amplitude on the nucleus can be taken as the sum of the amplitudes on the individual charges:

$$A = \sum_i A_i = f e^{i\vec{k}'\cdot\vec{r}} \sum_i e_i e^{iq\cdot\vec{\rho}_i}. \tag{12}$$

However, in quantum mechanics, $\vec{\rho}_i$ represent the position operators of the internal motion in the target. Therefore the last term should be replaced by the corresponding mean value in the ground state of the target. We define the form factor:

$$F(\vec{q}) = \frac{1}{Z_b e} < i| \sum_i e_i e^{i\vec{q}\cdot\vec{\rho}_i} |i>, \tag{13}$$

and then the cross section on an extended nucleus becomes

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{pl} |F(\vec{q})|^2, \tag{14}$$

where we identified the cross section on a pointlike particle as:

$$\left(\frac{d\sigma}{d\Omega}\right)_{pl} = (Z_b e)^2 |f|^2 \propto (Z_a Z_b e^2)^2. \tag{15}$$

The detailed and rigorous derivation of charge and magnetic FFs in a relativistic formalism is given in Section 4.

3.3. *The charge form factor*

Form factors are fundamental quantities, as they allow a direct comparison between the theory and the experiment. In order to determine $|F(\vec{q})|^2$ one has to measure the differential cross section, for different values of q. This can be done by varying the scattering angle and the energy of the projectile. If one wants to deduce the mean value of the charge density, in principle one can invert Eq. (13):

$$\rho(\vec{x}) =< \Psi_i|\hat{\rho}(\vec{x})|\Psi_i >= \frac{Z_b e}{(2\pi)^3}\int d^3 q F(\vec{q}) e^{-i\vec{q}\cdot\vec{x}}. \tag{16}$$

However, in practice, $F(\vec{q})$ can not be determined for all values of $\vec{q}$, due to the limits of the kinematically accessible region. Moreover, at large q, cross sections are very small and difficult to measure. Furthermore, the cross section is sensitive to the FF modulus squared, and does not give access to the phase. Therefore, in general, one assumes a specific mathematical function for $\rho(\vec{x})$, and free parameters that are fitted to the experimental data.

For small values of q^2 one can develop $F(q^2)$ in a Taylor series expansion on $\vec{q}\cdot\vec{x}$:

$$\begin{aligned} F(\vec{q}) &= \frac{1}{Z_b e}\int d^3\vec{x} e^{i\vec{q}\cdot\vec{x}}\rho(\vec{x}) \\ &= \frac{1}{Z_b e}\int d^3\vec{x}\left[1 + i\vec{q}\cdot\vec{x} - \frac{1}{2}(\vec{q}\cdot\vec{x})^2 + ...\right]\rho(\vec{x}) \\ &\simeq \frac{1}{Z_b e}\int_0^\infty x^2 dx \int_0^{2\pi} d\varphi \\ &\quad \int_{-1}^{1} d\cos\theta \left[1 + iqx\cos\theta - \frac{1}{2}q^2 x^2 \cos^2\theta\right]\rho(\vec{x}). \end{aligned}$$

The normalization is $\int_\Omega d^3\vec{x}\rho(\vec{x}) = Z_b e$. The second term does not give any contribution, as $\vec{q}\cdot\vec{x} = qx\cos\theta$ and $\int_{-1}^{1}\cos\theta d\cos\theta = 0$. This is a general fact, as x is a odd quantity, whereas $\rho(\vec{x})$, which contains the square of the wave function, is an even quantity with respect to space parity.

In case of spherical symmetry,

$$F(q) \sim 1 - \frac{1}{6}q^2 < r_c^2 > +O(q^2), \tag{17}$$

where we define the mean square root charge radius of the target, $< r_c^2 >$, as

$$< r_c^2 >= \frac{\int_0^\infty x^4\rho(x)dx}{\int_0^\infty x^2\rho(x)dx}.$$

3.3.1. *Application to different charge distributions*

Let us calculate $F(q)$ normalized to the full volume and charge:

$$F(q) = \frac{\int_\Omega d^3\vec{x} e^{i\vec{q}\cdot\vec{x}}\rho(\vec{x})}{\int_\Omega d^3\vec{x}\rho(\vec{x})}.$$

In case of spherical symmetry the denominator is:

$$D = 4\pi\int_0^\infty x^2\rho(x)dx$$

and the numerator:

$$N(q) = 2\pi\int_0^\infty x^2\rho(x)dx\int_{-1}^{1} d\cos\theta e^{iqx\cos\theta} = 2\pi\int_0^\infty x^2\rho(x)dx\frac{e^{iqx} - e^{-iqx}}{iqx}$$

Therefore:

$$F(q) = \frac{4\pi\int\frac{x}{q}\sin(qx)\rho(x)dx}{4\pi\int_0^\infty x^2\rho(x)dx}. \tag{18}$$

The typical shapes of charge density, with spherical symmetry, and the corresponding form factors and radii are shown in Table 1.

density $\rho(r)$	Form factor $F(q^2)$	r.m.s. $< r_c^2 >$	comments
δ	1	0	pointlike
e^{-ar}	$\frac{a^4}{(q^2+a^2)^2}$	$\frac{12}{a^2}$	dipole
$\frac{e^{-ar}}{r}$	$\frac{a^2}{q^2+a^2}$	$\frac{6}{a^2}$	monopole
$\frac{e^{-ar^2}}{r^2}$	$e^{-q^2/(4a^2)}$	$\frac{1}{2a}$	gaussian
ρ_0 for $x \leq R$ 0 for $r \geq R$	$\frac{3(\sin X - X\cos X)}{X^3}$ $X = qR$	$\frac{3}{5}R^2$	square well

As an example, let us calculate the radius corresponding to an exponential charge density, $\rho(x) = e^{-ax}$. First, we recall the following integrals:

$$\Gamma(\frac{1}{2}) = \int_0^\infty dz e^{-z} z^{-1/2} = \sqrt{\pi}, \tag{19}$$

$$\Gamma(x) = \int_0^\infty dz e^{-z} z^{x-1},\ \Gamma(x+1) = x\Gamma(x), \tag{20}$$

$$n! = \int_0^\infty dx x^n e^{-x} dx. \tag{21}$$

The radius is given by:

$$< r_c^2 >= \frac{\int_0^\infty x^4 e^{-ax} dx}{\int_0^\infty x^2 e^{-ax} dx} = \frac{a^{-5}\int_0^\infty (ax)^4 e^{-ax} d(ax)}{a^{-3}\int_0^\infty (ax) e^{-ax} d(ax)}$$

and the form factor:

$$F(q) = \frac{\frac{1}{q}\int_0^\infty x\sin(qx) e^{-ax} dx}{\int_0^\infty x^2 e^{-ax} dx}. \tag{22}$$

Applying (21), the denominator in Eq. (22) is:

$$D = \int_0^\infty x^2 e^{-ax} dx = \frac{2}{a^3}.$$

The numerator:

$$\begin{aligned} N &= \frac{1}{2iq}\int_0^\infty x(e^{iqx} - e^{-iqx}) e^{-ax} dx \\ &= \frac{1}{2iq}\int_0^\infty x\left[e^{-(-iq+a)x} - e^{-(iq+a)x}\right] dx \\ &= \frac{1}{2iq}\left[\frac{1}{(a-iq)^2}\int_0^\infty y e^{-y} dy - \frac{1}{(a+iq)^2}\int_0^\infty y e^{-y} dy\right]. \end{aligned}$$

Integrating per parts:

$$\Gamma(2) = \int y e^{-y} dy = -\int y d(e^{-y}) = -y e^{-y} + \int e^{-y} dy = -y e^{-y} - e^{-y}|_0^\infty = +1.$$

one finds:

$$N = \frac{1}{2iq}\left[\frac{1}{(a-iq)^2} - \frac{1}{(a+iq)^2}\right] = \frac{4aiq}{2iq(a^2+q^2)^2} = \frac{2a}{(a^2+q^2)^2}.$$

Finally:

$$F(q) = \frac{a^4}{(a^2+q^2)^2}.$$

Similarly one can verify all the results of Table 1.

3.3.2. *Units and orders of magnitudes*

The amplitude of the scattered wave is the sum of the amplitudes of the waves scattered from the individual constituents. An observer far from the target can see that the intensity of the scattered wave shows minima and maxima, as a function of the scattered angle, which correspond to interference among the different amplitudes A_i of the scattered waves. As in optics, one can introduce a resolving power δ:

$$\delta[fm] = \frac{\hbar}{|\vec{q}|} \sim \frac{200}{c|\vec{q}|}, \tag{23}$$

The quantity δ defines the spatial region that can be accessed in an experiment where the transferred momentum is $|\vec{q}|$. For example $|\vec{q}|$= 1 GeV (in center of mass system) in ep scattering corresponds to $\delta = 0.2$ fm.

Let us compare $\hbar c$ to the Bohr radius: $\lambda \sim 10^5$[fm]:

$$\frac{\hbar c}{\lambda} \simeq \frac{200[\text{Mev}]\,[\text{fm}]}{10^5[\text{fm}]} \simeq 2 \cdot 10^{-3}\text{MeV}. \tag{24}$$

3.4. *Extensions of the Rutherford Formula*

Let us summarize the assumptions under which the Rutherford formula holds:

- $U(r) = Z_1 Z_2 e^2/r$: coulomb interaction between target and projectile;
- validity of the Born approximation (lowest order/one photon exchange);
- non relativistic approximation;
- structureless and spinless particles.

The non relativistic approach is justified if the momenta of the particles are smaller than their masses ($p/m\Lambda\overline{\Lambda}1$). The differential cross section for spinless and pointlike+-*****- particles, in the relativistic case and in the Born approximation, was derived by N. F. Mott, including recoil effects of the target nucleus of mass M:[18]

$$\left(\frac{d\sigma}{d\Omega}\right)^{Lab}_{Mott} = \frac{e^2}{4E^2}\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\frac{1}{1+\dfrac{2E}{M}\sin^2(\theta/2)}. \tag{25}$$

In the language of Feynman diagrams, it is easy to verify the main features of the Mott cross section. The transition amplitude is proportional to $Z_i e$, the vertices contribution, which does not depend on the particle momenta for pointlike particles, and to the photon propagator $1/q^2$:

$$T_{fi} \propto \frac{Z_1 Z_2 e^2}{|\vec{q}|^2}, \ \left(\frac{d\sigma}{d\Omega}\right)^{Lab}_{Mott} \propto T^2_{fi}. \tag{26}$$

Further developments were given several years later. The extension of the Rutherford formula at the next order $\sim (Z\alpha)^{313}$ showed that the scattering of electrons and positrons is no more equivalent, because the correction depends on the charge:

$$\frac{d\sigma^{\pm}}{d\Omega} = \frac{d\sigma_R}{d\Omega}\left[1 \pm \pi\alpha Z \sin(\theta/2)\right], \ \frac{d\sigma_R}{dd\Omega} = \frac{(Z\alpha)^2}{4E^2\sin^4(\theta/2)}, \tag{27}$$

which leads to a charge asymmetry. Higher order corrections $\sim (Z\alpha)^n$ have been calculated more recently in the eikonal approximation[14,15,17] for charge asymmetry and polarization phenomena. A non trivial universal angular dependence is predicted, whose sign depends on the charge, observable in electron and positron scattering. The Rutherford cross section results modified by a factor:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\pm} \sim \left(\frac{d\sigma}{d\Omega}\right)_R \left[1 \pm \pi x \sin(\theta/2)\cos\varphi(x)\right], \ x = \frac{Z\alpha}{\beta}, \tag{28}$$

with

$$\Phi(x) = \cos\varphi(x) + i\sin\varphi(x) = \frac{\Gamma(\frac{1}{2}+ix)\Gamma(1-ix)}{\Gamma(\frac{1}{2}-ix)\Gamma(1+ix)}, \tag{29}$$

where β is the velocity v of the initial particle of mass m, in the Laboratory system, in units of c: $\beta = v/c = \sqrt{1-(4m^2)/E^2}$. Using the properties of Euler gamma function one obtains:

$$\varphi(x) = -4\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}c_n,$$
$$c_0 = \ln 2;\ c_1 = 3\xi_3;\ c_2 = 15\xi_5, ...,\ c_n = (2^{2n}-1)\xi_{2n+1},\ n \geq 1. \tag{30}$$

Applying the Stirling formula one can write

$$\cos(\varphi(x))|_{x>>1} \sim \frac{1}{4x}. \tag{31}$$

One can apply this formalism to the strong interaction, through the replacement $x = Z\alpha \to x_c = N\alpha_s$, proton and antiproton peripheral collisions on heavy nuclei may show a measurable multiphoton effect.

Further developments of the Rutherford formula include also high energy scattering on heavy targets (also in the eikonal approximation).[16]

3.5. *Cross section for a binary process*

The cross section σ for a binary process

$$a(p_1) + b(p_2) \to c(p_3) + d(p_4), \tag{32}$$

(where the momenta of the particles are indicated in parenthesis) characterizes the probability that a given process occurs. The number of events issued from a definite reaction is proportional to the number of incident particles N_B, the number of the target particles N_T and the constant of proportionality is the cross section:

$$N_F = \sigma N_a \times N_b. \tag{33}$$

The cross section can be viewed as an "effective area" over which the incident particle reacts. Therefore, its dimension is cm^2, but more often barn (1 barn=10^{-28} m^2), or fm^2 (1 fm=10^{-15} m) are used.

A useful quantity is the luminosity $\mathcal{L}$, defined as $\mathcal{L} = N_B$ [s^{-1}] N_T [cm^{-2}]. For simple counting estimations, $N_f = \sigma\mathcal{L}$. This is an operative definition, which is used in experimental physics.

On the other hand σ needs to be calculated theoretically for every type of process. The present derivation is done in a relativistic approach. This means that :
- The kinematics is relativistic,
- The matrix element $\mathcal{M}$, which contains the dynamics of the reaction is a relativistic invariant. In general it is function of kinematical variables, also relativistic $\mathcal{M} = f(s, t, u)$,
- σ has to be written in a relativistic invariant form.

The starting point is the following expression for the cross section

$$d\sigma = \frac{|\mathcal{M}|^2}{\mathcal{I}}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)d\mathcal{P}, \tag{34}$$

which is composed of four terms:

(1) The matrix element $\mathcal{M}$, which contains the dynamics of the reaction, and it is calculated following a model,
(2) The flux of colliding particles $\mathcal{I}$,
(3) The phase space for the final particles, $d\mathcal{P}$,
(4) A term which insures the conservation of the four-momentum $\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$ which is the product of four δ functions, because each component has to be conserved separately.

Let us calculate in detail each term.

3.5.1. *Definition of flux $\mathcal{I}$*

The flux is defined through the relative velocity of incoming and target particles:

$$\mathcal{I} = n_B n_T v_{rel}, \tag{35a}$$

$$\mathcal{I} = 4\sqrt{(p_1 \cdot p_2)^2 - M_1^2 M_2^2}, \tag{35b}$$

where $M_1(M_2)$ is the mass of the beam (target) particle, v_{rel} is the relative velocity between beam and target particles and the densities of the beam and target particles n_B, n_T are proportional to their energies as $n_i = 2E_i$.

Let us prove that the two expressions (35a) and (35b) are equivalent. It is more convenient to calculate $\mathcal{I}$ (Eq. 35) in the laboratory frame where the target is at rest:

$$p_1 = (E_1, \vec{p}_1), \quad p_2 = (M_2, 0), \quad |\vec{v}_{rel}| = |\vec{v}_1 - \vec{v}_2| = \frac{|\vec{p}_1|}{E_1} \ \Rightarrow\ n_B = 2E_1,\ n_T = 2M_2. \tag{36}$$

Replacing the equalities (36) in Eq. (35a):

$$\mathcal{I} = 2E_1 2M_2 \frac{|\vec{p}_1|}{E_1} = 4M_2|\vec{p}_1|$$

and in Eq. (35b) :

$$(p_1 \cdot p_2)^2 - M_1^2 M_2^2 = M_2^2 E_1^2 - M_1^2 M_2^2 = M_2^2 (E_1^2 - M_1^2) = M_2^2 |\boldsymbol{p}_1|^2, \text{ thus } \mathcal{I} = 4M_2|\vec{p}_1|$$

and the equalities (35) are proved. Moreover, we prove also that the flux does not depend on the reference frame, because it can be written in a Lorentz invariant form.

Let us consider the center of mass system (CMS):

$$p_1 = (E_1, \vec{k}),\ p_2 = (E_2, -\vec{k}),\ p_1 \cdot p_2 = E_1 E_2 + |\vec{k}|^2,\ M_1^2 = E_1^2 - |\vec{k}|^2,\ M_2^2 = E_2^2 - |\vec{k}|^2$$

and

$$\begin{aligned}(p_1 \cdot p_2)^2 - M_1^2 M_2^2 &= E_1^2 E_2^2 + 2E_1 E_2 |\vec{k}|^2 + |\vec{k}|^4 - E_1^2 E_2^2 + |\vec{k}|^2 (E_1^2 + E_2^2) - |\vec{k}|^4 \\ &= |\vec{k}|^2 (E_1 + E_2)^2 = |\vec{k}|^2 W^2. \end{aligned} \tag{37}$$

The flux, $\mathcal{I}$, can be written as

$$\mathcal{I} = 4|\vec{k}|W, \tag{38}$$

where $W = E_1 + E_2$ is the initial energy of the system in CMS.

3.5.2. *Phase space*

The phase space for a particle of energy E, mass M and four–momentum p (the number of states in the unit volume) can be written according to quantum mechanics in an invariant form:

$$d\mathcal{P} = \int \frac{d^4 p\ \delta(p^2 - M^2)}{(2\pi)^3} \Theta(E),$$

where the δ function insures that the particle is on mass shell and the step function $\Theta(E)$ insures that only the solution with positive energy is taken into account. Note that the wave functions of all particles entering in the matrix element must be normalized to one particle per unit volume. In this case all these wave functions contain the factor $1/\sqrt{2\varepsilon}$, where ε is the particle energy. Usually these factors are explicitly taken into account in the expression for the cross section, we insert them into the phase space.

Extracting the term which depends on energy:

$$d^4 p\ \delta(p^2 - M^2) = \delta^3 \vec{p} dE \delta(E^2 - \vec{p}^2 - M^2),$$

and using the property of the δ function

$$\int \delta[f(x)]dx = \sum \frac{1}{|f'(x_i)|}, \tag{39}$$

(x_i are the roots of $f(x)$), with $f(E) = E^2 - \vec{p}^2 - M^2$, and $f'(E) = 2E$ one finds:

$$\int dE \delta(E^2 - \vec{p}^2 - M^2)\Theta(E) = \frac{1}{2E}.$$

For the reaction under consideration:

$$d\mathcal{P} = \frac{d^3 \vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3 \vec{p}_4}{(2\pi)^3 2E_4}.$$

3.5.3. *Final formulas*

The total cross section can be written as:

$$\sigma = \frac{(2\pi)^4}{\mathcal{I}} \int |\mathcal{M}|^2 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \frac{d^3 \vec{p}_3}{(2\pi)^3 2E_3} \frac{d^3 \vec{p}_4}{(2\pi)^3 2E_4}. \tag{40}$$

One can see that it corresponds to a six-fold differential, but four δ functions are equivalent to four integrations. So finally, for a $2 \to 2$ process one is left with two independent variables, (E, θ) or (s, t). For three particles, one has nine differentials, four integrations, *i.e*., five independent variables.

The term $\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$ can be split into an energy and a space part: $\delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \delta(E_1 + E_2 - E_3 - E_4)\delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$.

Note that

$$\int \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)d^3\vec{p}_4 = 1 \tag{41}$$

in any reference frame.

Let us use spherical coordinates in CMS ($p_3 = (E_3, \vec{p})$, $p_4 = (E_4, -\vec{p})$, $d^3\vec{p} = |\vec{p}|^2 d\Omega dp$)and consider the quantity $\mathcal{J}$:

$$\mathcal{J} = \delta(E_1 + E_2 - E_3 - E_4)\frac{d^3\vec{p}_3}{4E_3E_4} = \delta(W - E_3 - E_4)\frac{|\vec{p}|^2 d\Omega dp}{4E_3E_4}, \tag{42}$$

where

$$E_3^2 = M_3^2 + |\vec{p}|^2,\ E_4^2 = M_4^2 + |\vec{p}|^2 \to E_3 dE_3 = E_4 dE_4 = |\vec{p}|dp.$$

After integration, using the property (39):

$$\mathcal{J} = \int \delta(W - E_3 - E_4)\frac{dE_3|\vec{p}|d\Omega}{4E_4} = \frac{|\vec{p}|d\Omega}{4E_4}\frac{1}{\left|\frac{d}{dE_3}(W - E_3 - E_4)\right|}, \tag{43}$$

where

$$\frac{d}{dE_3}(W - E_3 - E_4) = -1 - \frac{dE_4}{dE_3} = -1 - \frac{E_3}{E_4} = -\frac{W}{E_4} \tag{44}$$

and therefore

$$\mathcal{J} = \frac{|\vec{p}|d\Omega}{4W}. \tag{45}$$

Substituting Eqs. (38, 45) in Eq. (40) we find the general expression for the differential cross section of a binary process, in CMS:

$$\frac{d\sigma}{d\Omega} = \frac{\bar{|}\mathcal{M}|^2|\vec{p}|}{64\pi^2 W^2|\vec{k}|}, \tag{46}$$

and for the total cross section:

$$\sigma = \int \frac{|\mathcal{M}|^2|\vec{p}|}{64\pi^2 W^2|\vec{k}|}d\Omega. \tag{47}$$

In case of elastic scattering, $|\vec{k}| = |\vec{p}|$, therefore:

$$\frac{d\sigma}{d\Omega}^{el} = \frac{|\mathcal{M}|^2}{64\pi^2 W^2} = |\mathcal{F}^{el}|^2,\ \mathcal{F}^{el} = \frac{|\mathcal{M}|}{8\pi W}, \tag{48}$$

where $\mathcal{F}^{el}$ is the elastic amplitude.

3.6. *Reminder on the Dirac formalism*

Spin 1/2 particles

The elastic eN scattering involves four particles, with spin 1/2. The relativistic description of the spin properties of each of these particles is based on the Dirac equation:

$$(\hat{k} - m)u(k) = 0,\ \ \hat{k} = k_\mu\gamma_\mu = E\gamma_0 - \mathbf{k}\cdot\vec{\gamma},$$

where k is the particle four momentum ($k = (E, \mathbf{k})$) and $u(k)$ is a four-component Dirac spinor. We shall use the following representation of the Dirac 4×4 matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{49}$$

where $\vec{\sigma}$ is the standard set of the Pauli 2×2 matrices. On the basis of the Dirac equation one can write:

$$u(k) = \sqrt{E + m}\begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\mathbf{k}}{E + m}\chi \end{pmatrix}, \tag{50}$$

where χ is a two-component spinor. We used here the relativistic invariant normalization for the four-component spinor: $u^\dagger u = 2E$.

Spin 1/2 antiparticles

An antiparticle is described by the following spinor

$$v(k) = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\mathbf{k}}{E+m}\chi \\ \chi \end{pmatrix}. \tag{51}$$

The Dirac equation for particles (nucleon with momentum p_2) and antiparticles (antinucleon with momentum p_1) is:

$$\overline{u}(p_2)(\hat{p}_2 - m) = 0 \Rightarrow \overline{u}(p_2)\hat{p}_2 = \overline{u}(p_2)m,$$
$$(\hat{p}_1 + m)u(-p_1) = 0 \Rightarrow \hat{p}_1 u(-p_1) = -u(-p_1)m.$$

The density matrices $\rho = u(p)\overline{u}(p)$ for polarized and unpolarized particles and antiparticles are given in the Table 2. Applying the Dirac equation to the four-component spinor $u(p)$, of an electron with mass m_e, one can find the expressions for the density matrix of polarized electrons $\rho_{\alpha\beta} = u_\alpha(p)u^\dagger_\beta(p)$ reported in Table 2, where s_α is the four vector of the electron spin.

	particle	antiparticle
unpolarized	$\hat{p}+m$	$\hat{p}-m$
polarized	$(\hat{p}+m)\frac{1}{2}(1-\gamma_5\hat{s})$	$(\hat{p}-m)\frac{1}{2}(1-\gamma_5\hat{s})$

3.6.1. *Useful properties of Dirac matrices*

Some useful properties of Dirac matrices :

- The anticommutator is: $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor of the Minkowski space-time;
- $\hat{a}\hat{b} + \hat{b}\hat{a} = 2ab$, $\hat{a}\gamma_\mu + \gamma_\mu\hat{a} = 2a_\mu$, where a and b are four vectors;
- $Tr\ \gamma_\alpha\gamma_\beta = 4g_{\alpha\beta}$;
- $Tr\ \gamma_\alpha\gamma_\beta\gamma_\gamma = 0$;
- $Tr\ \gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta = 4\left(g_{\alpha\beta}\gamma_{\gamma\delta} + \gamma_{\beta\gamma}\gamma_{\delta\alpha} - \gamma_{\gamma\alpha}\gamma_{\delta\beta}\right)$.

3.6.2. *Relativistic formulation for the spin*

The four vector of the electron spin, s_α, satisfies the following two conditions:

$$s\cdot p = 0,\quad s^2 = -1. \tag{52}$$

In terms of the three-vector $\vec{\chi}$ of the electron polarization at rest, i.e., with zero three-momentum, the four-vector s can be written as:

$$s = \left(\frac{\vec{\chi}\cdot\mathbf{p}}{m_e}\right),\ \vec{\chi} + \frac{(\vec{\chi}\cdot\mathbf{p})\mathbf{p}}{m_e(\epsilon+m_e)}. \tag{53}$$

The condition $s^2 = -1$ corresponds to full electron polarization, so $s^2 = -|\vec{s}|^2 = -1$. Eq. (53) is simplified in case of relativistic electrons, $\epsilon \gg m_e$. In this case:

$$s_\alpha = \frac{\epsilon}{m}s_\ell(1,\mathbf{1}), \tag{54}$$

where $\mathbf{1}$ denotes the unit vector along $\mathbf{p}$ and $s_\ell = \vec{\chi}\cdot\mathbf{p}/|\mathbf{p}| \equiv \lambda$.

Taking into account that for relativistic electrons:

$$p_\alpha = \epsilon(1,\mathbf{1}), \tag{55}$$

it is possible to re-write Eq. (54) in the form:

$$s_\alpha = \frac{p_{1\alpha}}{m}\lambda. \tag{56}$$

One can find the following expression for the density matrix of a relativistic polarized electron:

$$\begin{aligned}\rho &= \frac{1}{2}(\hat{p}+m_e)\left(1-\gamma_5\frac{\hat{p}}{m_e}\lambda\right) = \frac{1}{2}(\hat{p}+m_e)+\frac{\lambda}{2}(\hat{p}+m_e)\frac{\hat{p}}{m_e}\gamma_5 \\ &= \frac{1}{2}(\hat{p}+m_e)+\frac{\lambda}{2}\left(p^2+m_e\hat{p}\right)\frac{1}{m_e}\gamma_5 \\ &= \frac{1}{2}(\hat{p}+m_e)(1+\lambda\gamma_5) \equiv \frac{1}{2}\hat{p}(1+\lambda\gamma_5),\end{aligned} \tag{57}$$

where we used the following property of the γ_5-matrix: $\hat{p}\gamma_5+\gamma_5\hat{p}=0$, for any four-vector p_α.

4. Relativistic formalism for ep elastic scattering

Let us derive step by step the elastic cross section and the polarization observables for electron proton scattering, in the Born approximation, in a fully relativistic formalism, taking into account that the proton has a spin and an internal structure. This derivation closely follows lecture notes earlier prepared with Prof. M. P. Rekalo.[42]

4.1. *Relativistic kinematics*

The Feynman diagram for elastic *eN*-scattering is shown in Fig. 2, assuming one-photon exchange. The notations of the particle four-momenta are also shown in the Fig. 2 and in Table 3 (we will use in our calculation the system where ℏ=c=1).

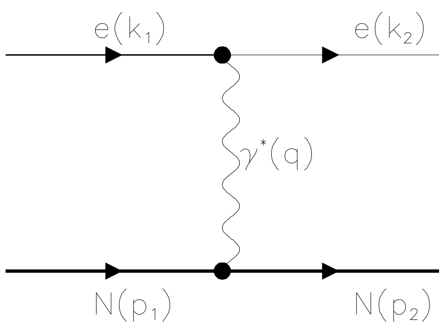

Fig. 2. One-photon exchange diagram for elastic scattering, $e+N\to e+N$.

The conservation of four-momenta at each vertex of the diagram can be written as:

$$q = k_1 - k_2 = p_2 - p_1, \tag{58}$$

which is valid in any reference frame. Using the relation (58) in the Lab-system, we derive the formula for the momentum transfer squared q^2, which is the basic kinematical variable for elastic *eN* scattering:

$$q^2 = (p_2-p_1)^2 = p_1^2+p_2^2-2M_pE_2 = 2M_p^2-2M_pE_2 = -2M_pT,$$

where E_2 is the total energy of the final nucleon, M_p is the nucleon mass, and $T=E_2-M_p$ is the kinetic energy. This formula demonstrates that, for elastic scattering, the momentum transfer squared, q^2, is negative for all energies and scattering angles of the outgoing electron. As q^2 is a relativistic invariant, this is true in any reference system. The kinematical region for which $q^2<0$ is called the $Space-Like$ region.

	Lab	**CMS**	**Breit**
q	$(\omega,\mathbf{q})$	$(\widetilde{\omega},\widetilde{\mathbf{q}})$	$(\omega_B=0,\mathbf{q_B})$
k_1	$(\epsilon_1,\mathbf{k_1})$	$(\widetilde{\epsilon_1},\widetilde{\mathbf{k_1}})$	$(\epsilon_{1B},\mathbf{k_{1B}})$
p_1	$(M_p,0)$	$(\widetilde{E_1},-\widetilde{\mathbf{k_1}})$	$(E_{1B},\mathbf{p_{1B}})$
k_2	$(\epsilon_2,\mathbf{k_2})$	$(\widetilde{\epsilon_2},\widetilde{\mathbf{k_2}})$	$(\epsilon_{2B},\mathbf{k_{2B}})$
p_2	$(E_2,\mathbf{p_2})$	$(\widetilde{E_2},-\widetilde{\mathbf{k_2}})$	$(E_{2B},-\mathbf{p_{1B}})$

4.1.1. *Proton kinematics in the Breit system*

The most convenient frame for the analysis of elastic eN-scattering is the Breit frame, which is defined as the system where the initial and final nucleon energies are the same. As a consequence, the energy of the virtual photon vanishes and its four-momentum squared, q^2, coincides with its three-momentum squared, $\mathbf{q_B}^2$, more exactly, $q^2 = -\mathbf{q_B}^2$. The derivation of the formalism in Breit system is therefore simpler and has some analogy with a non-relativistic description of the nucleon electromagnetic structure. From the energy conservation, and from the definition of the Breit system, one can find:

$$\omega_B = E_{1B} - E_{2B} = 0,$$

where all kinematical quantities in the Breit system are denoted with subscript B. The proton three-momentum can be found from the relation

$$E_{1B}^2 = E_{2B}^2 = \mathbf{p_{1B}}^2 + M_p^2 = \mathbf{p_{2B}}^2 + M_p^2, \ i.e., \ \mathbf{p_{1B}}^2 = \mathbf{p_{2B}}^2.$$

The physical solution of this quadratic relation is $\mathbf{p_{1B}} = -\mathbf{p_{2B}}$, as the Breit system moves in the direction of the outgoing proton. From the three-momentum conservation, in the Breit system $\mathbf{q_B} + \mathbf{p_{1B}} = \mathbf{p_{2B}}$, one can find:

$$\mathbf{p_{1B}} = -\frac{\mathbf{q_B}}{2}, \ \ \mathbf{p_{2B}} = \frac{\mathbf{q_B}}{2}.$$

The proton energy can be expressed as a function of $\mathbf{q_B}^2$, and therefore of q^2:

$$E_{1B}^2 = E_{2B}^2 = M_p^2 + \frac{\mathbf{q_B}^2}{4} = M_p^2 - \frac{q^2}{4} = M_p^2(1+\tau),$$

where we replaced the three-momentum in Breit system by the four-momentum and we introduced the dimensionless quantity $\tau = \dfrac{Q^2}{4M_p^2} = -\dfrac{q^2}{4M_p^2} \geq 0$.

4.1.2. *Electron kinematics in the Breit system*

The conservation of the four momentum, at the electron vertex, can be written, in any reference system, as: $k_1 = q + k_2$ (the virtual photon is radiated by the electron). In the Breit system, the energy and momentum conservation is:

$$\begin{cases} \epsilon_{1B} = \ \omega_B + \epsilon_{2B} = \epsilon_{2B}, \\ \mathbf{k_{1B}} = \mathbf{q_B} + \mathbf{k_{2B}}. \end{cases} \tag{59}$$

In order to proceed, we must define a reference (coordinate) system: we choose the z-axis parallel to the photon three-momentum in the Breit system: $z \parallel \mathbf{q_B}$, and the xz-plane as the scattering plane. So we can write:

$$\begin{cases} \epsilon_{1B}^2 = \epsilon_{2B}^2 = m_e^2 + (k_{1B}^x)^2 + (k_{1B}^z)^2 = m_e^2 + (k_{2B}^x)^2 + (k_{2B}^z)^2, \\ k_{1B}^x = k_{2B}^x, \\ k_{1B}^y = k_{2B}^y = 0, \\ k_{1B}^z = q_B + k_{2B}^z. \end{cases} \tag{60}$$

It follows $k_{1B}^z = -k_{2B}^z = \dfrac{q_B}{2}$ (the other possible solution $k_{1B}^z = k_{2B}^z$ would imply $q_B = 0$). A graphical representation for the conservation of three-momenta is given in Fig. 3.

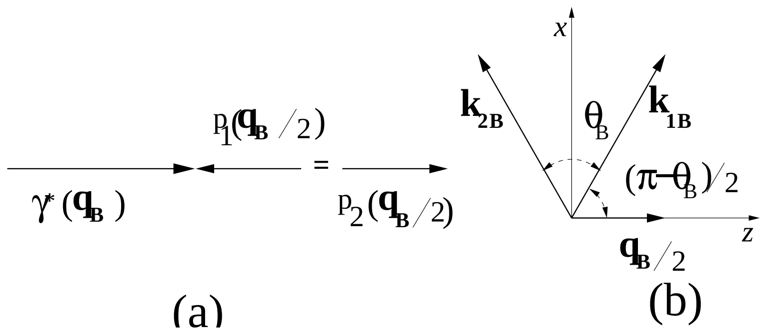

Fig. 3. Proton (a) and electron (b) three-momenta representation for elastic eN-scattering in the Breit system.

Then we can write, for the components of the initial and final electron three-momenta:

$$\mathbf{k_{1B}} = (k^x_{1B}, k^y_{1B}, k^z_{1B}) = \left(\frac{q_B}{2}\cot\frac{\theta_B}{2}, 0, \frac{q_B}{2}\right) = \frac{\sqrt{-q^2}}{2}\left(\cot\frac{\theta_B}{2}, 0, 1\right), \tag{61}$$

$$\mathbf{k_{2B}} = (k^x_{2B}, k^y_{2B}, k^z_{2B}) = \left(\frac{q_B}{2}\cot\frac{\theta_B}{2}, 0, -\frac{q_B}{2}\right) = \frac{\sqrt{-q^2}}{2}\left(\cot\frac{\theta_B}{2}, 0, -1\right). \tag{62}$$

The energy of the electron (neglecting the electron mass) is given by:

$$\epsilon^2_{1B} = \mathbf{k_{1B}}^2 = (k^x_{1B})^2 + (k^z_{1B})^2 = \frac{-q^2}{4\sin^2\frac{\theta_B}{2}} \text{ and } \epsilon_{2B} = \epsilon_{1B}.$$

4.1.3. *Relation between the electron scattering angles in the Lab system, θ_e and in the Breit system, θ_B*

As the Breit system is moving along the z-axis, the x and y components of the particle three-momenta do not change after transformation from the Lab to the Breit system:

$$\begin{cases} k^B_{1y} = k_{2y} = 0, \\ k^B_{1x} = k_{1x}. \end{cases} \tag{63}$$

From $\mathbf{k_1}^2 = k^2_{1x} + k^2_{1z}$ one can find:

$$k^2_{1x} = \mathbf{k_1}^2 - \frac{(\mathbf{k_1}\cdot\mathbf{q})^2}{\mathbf{q}^2} = \frac{\mathbf{k_1}^2\mathbf{q}^2 - (\mathbf{k_1}\cdot\mathbf{q})^2}{\mathbf{q}^2} = \frac{\epsilon_1^2\epsilon_2^2\sin^2\theta_e}{\mathbf{q}^2} = \frac{4\epsilon_1^2\epsilon_2^2}{\mathbf{q}^2}\sin^2\frac{\theta_e}{2}\cos^2\frac{\theta_e}{2}, \tag{64}$$

where we replaced $\mathbf{q} = \mathbf{k_1} - \mathbf{k_2}$, $\mathbf{k_1}^2 = \epsilon_1^2$, $\mathbf{k_2}^2 = \epsilon_2^2$ after setting $m_e = 0$. On the other hand we find for q^2 the following expression in the Lab system (in terms of the energies of the initial and final electron and of the electron scattering angle):

$$\begin{aligned} q^2 &= (k_1 - k_2)^2 = 2m_e^2 - 2k_1\cdot k_2 \overset{m_e=0}{\simeq} -2\epsilon_1\epsilon_2 + 2\mathbf{k_1}\cdot\mathbf{k_2} = -2\epsilon_1\epsilon_2(1-\cos\theta_e) \\ &= -4\epsilon_1\epsilon_2\sin^2\frac{\theta_e}{2}. \end{aligned} \tag{65}$$

Comparing Eqs. (64) and (65), we find:

$$k^2_{1x} = \frac{(q^2)^2}{4\mathbf{q}^2}\cot^2\frac{\theta_e}{2}.$$

Using the relations: $\mathbf{q}^2 = \omega^2 - q^2$ and $q^2 + 2q\cdot p_1 + p_1^2 = p_2^2$, we have, in the Lab system, $\omega = -\frac{q^2}{2m}$ and $\mathbf{q}^2 = -q^2(1+\tau)$. Finally:

$$k^2_{1x} = -\frac{q^2}{4(1+\tau)}\cot^2\frac{\theta_e}{2}.$$

So, from the relation $k^2_{1x} = (k^x_{1B})^2$, we find the following relation between the electron scattering angle in the Lab system and in the Breit system:

$$\boxed{\cot^2\frac{\theta_B}{2} = \frac{\cot^2\theta_e/2}{1+\tau}.} \tag{66}$$

4.1.4. *Expression of* $\sin\frac{\theta_B}{2}$ *in terms of energies in the Lab system*

Let us find the expression for $\sin\frac{\theta_B}{2}$ in terms of the kinematical variables in the Lab-system.

Using the relation (66), one finds:

$$\frac{1}{\sin^2\frac{\theta_B}{2}} = 1 + \frac{\cot^2\frac{\theta_e}{2}}{1+\tau} = \frac{1}{1+\tau}\left[\tau + \frac{1}{\sin^2\frac{\theta_e}{2}}\right] = \frac{1}{1+\tau}\frac{1+\tau\sin^2\frac{\theta_e}{2}}{\sin^2\frac{\theta_e}{2}} \tag{67}$$

So

$$1+\tau\sin^2\frac{\theta_e}{2} = 1+\frac{\frac{\epsilon_1^2}{M_p^2}\sin^4\frac{\theta_e}{2}}{1+2\frac{\epsilon_1}{M_p}\sin^2\frac{\theta_e}{2}} = \frac{(1+\frac{\epsilon_1}{M_p}\sin^2\frac{\theta_e}{2})^2}{1+2\frac{\epsilon_1}{M_p}\sin^2\frac{\theta_e}{2}}. \tag{68}$$

Using the relation (87) between the initial and final electron energy, we have:

$$1+\frac{\epsilon_1}{M_p}\sin^2\frac{\theta_e}{2} = \frac{1}{2}\frac{\epsilon_1+\epsilon_2}{\epsilon_2}. \tag{69}$$

Substituting (69) in (67), one finally finds:

$$\frac{1}{\sin^2\frac{\theta_B}{2}} = \frac{(\epsilon_1+\epsilon_2)^2}{(-q^2)(1+\tau)}. \tag{70}$$

4.2. *Dynamics*

Electron proton scattering through one photon exchange is illustrated by the Feynman diagram in Fig. 2, which includes two vertexes: (1) the electron vertex, which is described by QED-rules, (2) the proton vertex described by QCD and hadron electrodynamics, connected by the virtual photon line. The matrix element corresponding to this diagram, is written as:

$$\mathcal{M} = \frac{e^2}{q^2}\ell_\mu \mathcal{J}_\mu = \frac{e^2}{q^2}\ell\cdot\mathcal{J}, \tag{71}$$

where $\ell_\mu = \overline{u}(k_2)\gamma_\mu u(k_1)$ is the electromagnetic current of electron. The nucleon electromagnetic current, $\mathcal{J}_\mu$ describes the proton vertex and is generally written in terms of Pauli and Dirac FFs F_1 and F_2:

$$\mathcal{J}_\mu = \overline{u}(p_2)\left[F_1(q^2)\gamma_\mu - \frac{\sigma_{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p_1), \tag{72}$$

with

$$\sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{2}.$$

Note that $\mathcal{J}\cdot q = 0$, for any values of F_1 and F_2, i.e., the current $\mathcal{J}_\mu$ is conserved[†].

Using the Dirac equation for the four-component spinors of the initial and final nucleon, Eq. (72) can be rewritten in a simpler form, using:

$$\overline{u}(p_2)\frac{\sigma_{\mu\nu}q_\nu}{2M_p}u(p_1) = \overline{u}(p_2)\left[\gamma_\mu - \frac{(p_1+p_2)_\mu}{2M_p}\right]u(p_1). \tag{73}$$

which is also conserved.

[†]This can be easily proved as follows. The term $\sigma_{\mu\nu}q_\mu q_\nu$ vanishes, because it is the product of a symmetrical and antisymmetrical tensors, and $\overline{u}(p_2)\hat{q}u(p_1) = \overline{u}(p_2)(\hat{p}_2-\hat{p}_1)u(p_1) = \overline{u}(p_2)(M_p-M_p)u(p_1) = 0$, as a result of the Dirac equation for both four-component spinors, $u(p_1)$ and $u(p_2)$. Note that the current (72) is conserved only when both nucleons (in initial and final states) are real, the form factor F_1 violates the current conservation, if one nucleon is virtual.

Let us prove Eq. (73).
Using the definition for $\sigma_{\mu\nu}$, one can write:

$$\bar{u}(p_2)\frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{4M_p}q_\nu u(p_1) = \bar{u}(p_2)\frac{\gamma_\mu\hat{q} - \hat{q}\gamma_\mu}{4M_p}u(p_1).$$

Recalling that $q = p_2 - p_1$ with $\hat{a} = a_\mu\gamma_\mu$:

$$\bar{u}(p_2)\frac{\gamma_\mu(\hat{p_2} - \hat{p_1}) - (\hat{p_2} - \hat{p_1})\gamma_\mu}{4M_p}u(p_1).$$

Applying the Dirac equation:

$$(\hat{p} - M_p)u(p) = 0 \to \hat{p}u(p) = M_p u(p),$$
$$\bar{u}(p)(\hat{p} - M_p) = 0 \to \bar{u}(p)\hat{p} = \bar{u}(p)M_p,$$

we find:

$$\bar{u}(p_2)\frac{\gamma_\mu(\hat{p_2} - M_p) - (M_p - \hat{p_1})\gamma_\mu}{4M_p}u(p_1) = -\frac{1}{2}\bar{u}(p_2)\gamma_\mu u(p_1) + \frac{1}{4M_p}\bar{u}(p_2)\left[\gamma_\mu\hat{p_2} + \hat{p_1}\gamma_\mu\right]u(p_1). \tag{74}$$

Using the properties: $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$, $\hat{a}\hat{b} + \hat{b}\hat{a} = 2ab$, $\hat{a}\gamma_\mu + \gamma_\mu\hat{a} = 2a_\mu$ we have $\hat{p_1}\gamma_\mu = -\gamma_\mu\hat{p_1} + 2p_{1\mu}$, so that:

$$\begin{aligned}\frac{1}{4M_p}\bar{u}(p_2)\left[\gamma_\mu\hat{p_2} + \hat{p_1}\gamma_\mu\right]u(p_1) &= \frac{1}{4M_p}\bar{u}(p_2)\left[-\hat{p_2}\gamma_\mu + 2p_{2\mu} - \gamma_\mu\hat{p_1} + 2p_{1\mu}\right]u(p_1)\\ &= \frac{1}{4M_p}\bar{u}(p_2)\left[-2\gamma_\mu M_p + 2(p_{2\mu} + p_{1\mu})\right]u(p_1)\\ &= \frac{1}{2}\bar{u}(p_2)\left[-\gamma_\mu + \frac{(p_{2\mu} + p_{1\mu})}{M_p}\right]u(p_1).\end{aligned} \tag{75}$$

Inserting Eq. (75) in (74), we find Eq. (73).

Note that the relation (73) is correct only when both nucleons are on mass shell, i.e; they are described by the four-component spinors $u(p)$, satisfying the Dirac equation. It is not the case for the quasi-elastic scattering of electrons by atomic nuclei, $e + A \to e + p + X$, which contains as subprocess the scattering $e + N^* \to e + N$, where N^* is a virtual nucleon.

Eq. (73) is an expression of the nucleon electromagnetic current, which holds in any reference system. However, for the analysis of polarization phenomena, the Breit system is the most preferable. First of all, the explicit expression of the current $\mathcal{J}_\mu = (\mathcal{J}_0, \vec{\mathcal{J}})$ is simplified in the Breit system:

$$\begin{cases}\mathcal{J}_0 = \bar{u}(p_2)\left[(F_1 + F_2)\gamma_0 - \dfrac{(E_{1B} + E_{2B})}{2M_p}F_2\right]u(p_1),\ E_{1B} = E_{2B} = E,\\ \vec{\mathcal{J}} = \bar{u}(p_2)\left[(F_1 + F_2)\vec{\gamma} - \dfrac{(\mathbf{p_{1B}} + \mathbf{p_{2B}})}{2M_p}F_2\right]u(p_1) = (F_1 + F_2)\bar{u}(p_2)\vec{\gamma}u(p_1).\end{cases} \tag{76}$$

With $u(p_1)$ and $u(p_2)$ defined according to (50) we find, for the time component $\mathcal{J}_0$ of the current $\mathcal{J}_\mu$:

$$\begin{aligned}\mathcal{J}_0 &= (F_1 + F_2)u^\dagger(p_2)u(p_1) - F_2\frac{E}{M_p}u^\dagger(p_2)\gamma_0 u(p_1)\\ &= (E + M_p)\left\{(F_1 + F_2)\chi_2^\dagger\left(1\,,\ \frac{\vec{\sigma}\cdot\mathbf{q_B}}{2(E + M_p)}\right)\begin{pmatrix}\chi_1\\ \dfrac{-\vec{\sigma}\cdot\mathbf{q_B}}{2(E + M_p)}\chi_1\end{pmatrix}\right.\\ &\left.-F_2\frac{E}{M_p}\chi_2^\dagger\left(1\,,\ \frac{\vec{\sigma}\cdot\mathbf{q_B}}{2(E + M_p)}\right)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}\chi_1\\ \dfrac{-\vec{\sigma}\cdot\mathbf{q_B}}{2(E + M_p)}\chi_1\end{pmatrix}\right\} =\\ &= 2M_p\chi_2^\dagger\chi_1\left(F_1 - \tau F_2\right),\end{aligned} \tag{77}$$

where we used the definition:

$$\mathbf{p_{2B}}^2 = E^2 - M_p^2 = \frac{\mathbf{q_B}^2}{4},\ \text{so that}\ \frac{\mathbf{q_B}^2}{4(E + M_p)^2} = \frac{E - M_p}{E + M_p},$$

and

$$\bar{u}(p_2) = u^\dagger(p_2)\gamma_0,\ \gamma_0^2 = 1 \text{ and } (\vec{\sigma}\cdot\mathbf{q})(\vec{\sigma}\cdot\mathbf{q}) = \mathbf{q}^2.$$

For the vector part $\vec{\mathcal{J}}$ of the nucleon electromagnetic current we can find similarly:

$$\vec{\mathcal{J}} = (F_1 + F_2)(E + M_p)\chi_2^\dagger \left(1\,,\ -\frac{\vec{\sigma}\cdot\mathbf{q_B}}{2(E+M_p)}\right)\begin{bmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{bmatrix}\begin{pmatrix}\chi_1\\ \dfrac{-\vec{\sigma}\cdot\mathbf{q_B}}{2(E+M_p)}\chi_1\end{pmatrix}$$
$$= -\frac{1}{2}(F_1+F_2)\chi_2^\dagger(\vec{\sigma}\vec{\sigma}\cdot\mathbf{q_B} - \vec{\sigma}\cdot\mathbf{q_B}\vec{\sigma})\chi_1$$

Multiplying the left and right side by $\vec{a}$:

$$\vec{\mathcal{J}}\cdot\vec{a} = \vec{\sigma}\cdot\vec{a}\vec{\sigma}\cdot\mathbf{q_B} - \vec{\sigma}\cdot\mathbf{q_B}\vec{\sigma}\cdot\vec{a}$$
$$= \vec{a}\cdot\mathbf{q_B} + i\vec{\sigma}\cdot(\vec{a}\times\mathbf{q_B}) - (\mathbf{q_B}\cdot\vec{a} + i\vec{\sigma}\cdot\mathbf{q_B}\times\vec{a}) = -2i\vec{a}\cdot\vec{\sigma}\times\mathbf{q_B}.$$

Finally:

$$\boxed{\begin{aligned}\mathcal{J}_0 &= 2M_p\chi_2^\dagger\chi_1(F_1 - \tau F_2)\,,\\ \vec{\mathcal{J}} &= i\chi_2^\dagger\vec{\sigma}\times\mathbf{q_B}\chi_1(F_1+F_2)\,.\end{aligned}}$$

These expressions for the different components of the current $\mathcal{J}_\mu$ are valid in the Breit frame only, and allow to introduce in a straightforward way the Sachs nucleon electromagnetic FFs,[43] electric and magnetic, which are related to F_1 and F_2 as in Table 4. Note that, by convention, $\tau > 0$ is chosen to be always positive. In TL region, these relations are correct after replacement $\tau \to -\tau$.

$G_M = F_1 + F_2$	$F_1 = \dfrac{G_E + \tau G_M}{1+\tau}$
$G_E = F_1 - \tau F_2$	$F_2 = \dfrac{G_M - G_E}{1+\tau}$

Such identification can be easily understood, if one takes into account that the time component of the current, $\mathcal{J}_0$, describes the interaction of the nucleon electric charge with the Coulomb potential. Correspondingly, the space component $\vec{\mathcal{J}}$ describes the interaction of the nucleon spin with the magnetic field.

4.3. *The unpolarized cross section*

The starting point is the expression (34) for the cross section. From Eq.(71) we can find the following representation for $\overline{|\mathcal{M}|}$ (the bar denotes the averaging over the polarizations of the initial electron and the summing over the polarizations of the final electrons):

$$\overline{|\mathcal{M}|^2} = \left(\frac{e^2}{q^2}\right)^2 \overline{|\ell\cdot\mathcal{J}|^2} = \left(\frac{e^2}{q^2}\right)^2 L_{\mu\nu}W_{\mu\nu}, \tag{78}$$

where:

$L_{\mu\nu} = \overline{\ell_\mu\ell_\nu^*}$ is the leptonic tensor,

$W_{\mu\nu} = \overline{\mathcal{J}_\mu\mathcal{J}_\nu^*}$ is the hadronic tensor.

The product of the tensors $L_{\mu\nu}$ and $W_{\mu\nu}$ is a relativistic invariant, therefore it can be calculated in any reference system. The differential cross section, in any coordinate system, can be expressed in terms of the matrix element as:

$$d\sigma = \frac{(2\pi)^4\overline{|\mathcal{M}|^2}}{4\sqrt{(k_1\cdot p_1)^2 - m_e^2M_p^2}}\delta^4(k_1 + p_1 - k_2 - p_2)\frac{d^3\mathbf{k_2}}{(2\pi)^3 2\epsilon_2}\frac{d^3\mathbf{p_2}}{(2\pi)^3 2E_2}. \tag{79}$$

To compare with experiments, it is more convenient to use the differential cross section in Lab system, $d\sigma/d\Omega_e$, where $d\Omega_e$ is the element of the electron solid angle in the Lab system. This can be done, integrating Eq. (79), using the properties of the δ^4 function.

First of all, let us integrate over the three-momentum $\mathbf{p_2}$, applying the three momentum conservation for the considered process:

$$\int d^3\mathbf{p_2}\delta^3(\mathbf{k_1}-\mathbf{k_2}-\mathbf{p_2})=1, \text{ with the condition } \mathbf{p_2}=\mathbf{k_1}-\mathbf{k_2}.$$

Using the definition $d^3\mathbf{k_2} \overset{m_e=0}{=} d\Omega_e \mathbf{k_2}^2 d|\mathbf{k_2}| \simeq d\Omega_e \epsilon_2^2 d\epsilon_2$, we can integrate over the electron energy, taking into account the conservation of energy:

$$\delta(\epsilon_1+M_p-\epsilon_2-E_2)d\epsilon_2=\delta\left(\epsilon_1+M_p-\epsilon_2-\sqrt{M_p^2+\mathbf{p_2}^2}\right)d\epsilon_2=$$

$$\delta\left(\epsilon_1+M_p-\epsilon_2-\sqrt{M_p^2+(\mathbf{k_1}-\mathbf{k_2})^2}\right)d\epsilon_2.$$

Let us recall that:

$$\int \delta\left[f(\epsilon_2)\right]d\epsilon_2=\frac{1}{|f'(\epsilon_2)|},$$

where $f(\epsilon_2)=\epsilon_1+M_p-\epsilon_2-\sqrt{M_p^2+\epsilon_1^2+\epsilon_2^2-2\epsilon_1\epsilon_2\cos\theta_e}$. Therefore:

$$|f'(\epsilon_2)|=1+\frac{\epsilon_2-\epsilon_1\cos\theta_e}{E_2}=1+\frac{\epsilon_2^2-\mathbf{k_1}\cdot\mathbf{k_2}}{\epsilon_2E_2}=\frac{k_2\cdot(k_1+p_1)}{\epsilon_2E_2},$$

where we multiplied by ϵ_2 the numerator and denominator, and we used the conservation of energy $\epsilon_2+E_2=\epsilon_1+M_p$. But from the conservation of four-momentum, in the following form $k_1+p_1-k_2=p_2$, we have:

$$(k_1+p_1)^2+k_2^2-2(k_1+p_1)\cdot k_2=M_p^2.$$

So $2(k_1+p_1)\cdot k_2=(k_1+p_1)^2-M_p^2=2k_1\cdot p_1=2\epsilon_1M_p$ (in Lab system). Finally

$$|f'(\epsilon_2)|=\frac{\epsilon_1}{\epsilon_2}\frac{M_p}{E_2}.$$

After substituting in Eq. (79), one finds the following relation between $\overline{|\mathcal{M}|^2}$ and the differential cross section in Lab system:

$$\frac{d\sigma}{d\Omega_e}=\frac{\overline{|\mathcal{M}|^2}}{64\pi^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\frac{1}{M_p^2}. \tag{80}$$

4.3.1. *Hadronic tensor $W_{\mu\nu}$*

Let us calculate the hadronic tensor $W_{\mu\nu}$ in the Breit system, where there is a simple expression of the nucleon current. Let us write this current as: $\mathcal{J}_\mu=\chi_2^\dagger F_\mu\chi_1$, with $F_\mu=2M_pG_E$, for $\mu=0$ and $F_\mu=i\vec{\sigma}\times\mathbf{q_B}G_M$, for $\mu=x,y,z$. So the the four components of F_μ, in terms of FFs G_E and G_E, can be written as:

$$F_\mu=\begin{cases}2M_pG_E & ,\mu=0\\ i\sqrt{-q^2}G_M\sigma_y & ,\mu=x\\ -i\sqrt{-q^2}G_M\sigma_x & ,\mu=y\\ 0 & ,\mu=z.\end{cases} \tag{81}$$

Therefore, the hadronic tensor $W_{\mu\nu}$ can be written as follows:

$$W_{\mu\nu}=\overline{(\chi_2^\dagger F_\mu\chi_1)(\chi_1^\dagger F_\nu^\dagger\chi_2)}=\frac{1}{2}TrF_\mu\rho_1F_\nu^\dagger\rho_2,$$

where the averaging (summing) acts only on the two-component spinors, and we introduced density matrix for the nucleon: $\rho=\chi\chi^\dagger$, $\rho_{ab}=\chi_a\chi_b^*$, and $a,b=1,2$ are the spinor indexes. We included the statistical factor $1/(2s+1)=1/2$, for the initial nucleon.

In case of unpolarized particles $\rho=1/2$, and

$$W_{\mu\nu}=\frac{1}{2}TrF_\mu F_\nu^\dagger.$$

4.3.2. *Leptonic tensor $L_{\mu\nu}$*

The leptonic tensor, which describes the electron vertex, is written as:

$$L_{\mu\nu} = \overline{\ell_\mu \ell_\nu^*} = \overline{\overline{u}(k_2)\gamma_\mu u(k_1)\left[\overline{u}(k_2)\gamma_\nu u(k_1)\right]^*}.$$

Recalling that

$$\overline{u} = u^\dagger \gamma_0, \quad \overline{u}^\dagger = (u^\dagger\gamma_0)^\dagger = \gamma_0^\dagger u = \gamma_0 u, \quad \gamma_0\gamma_0 = 1, \; \gamma_0^\dagger = \gamma_0,$$

we can write:

$$\begin{aligned} L_{\mu\nu} &= \overline{\overline{u}(k_2)\gamma_\mu u(k_1)u^\dagger(k_1)\gamma_\nu^\dagger \overline{u}(k_2)} = \overline{\overline{u}(k_2)\gamma_\mu u(k_1)u^\dagger(k_1)\gamma_0\gamma_0\gamma_\nu^\dagger\gamma_0 u(k_2)} \\ &= \overline{\overline{u}(k_2)\gamma_\mu u(k_1)\overline{u}(k_1)\gamma_0\gamma_\nu^\dagger\gamma_0 u(k_2)} = \frac{1}{2} Tr\gamma_\mu\rho_e^1\gamma_\nu\rho_e^2. \end{aligned} \tag{82}$$

From the Dirac theory we can write: $u(k)\overline{u}(k) = \hat{k} + m_e = \rho$:

$$L_{\mu\nu} = \frac{1}{2} Tr\gamma_\mu(\hat{k_1} + m_e)\gamma_\nu(\hat{k_2} + m_e) = Tr\gamma_\mu\hat{k_1}\gamma_\nu\hat{k_2} + m_e^2 Tr\gamma_\mu\gamma_\nu.$$

Recalling that $Tr\gamma_a\gamma_b = 4g_{ab}$ ($g_{ab} = 1$, for $a, b = 0$, $g_{ab} = -1$, for $a, b = x, y$, or z; and $Tr\gamma_a\gamma_b\gamma_c\gamma_d = 4(g_{ab}g_{cd} + g_{bc}g_{da} - g_{ac}g_{bd})$ we derive :

$$L_{\mu\nu} = 2k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{2\mu} + 2g_{\mu\nu}(m_e^2 - k_1 \cdot k_2).$$

Using that $k_1 = q + k_2$; $q^2 = 2(m_e^2 - k_1 \cdot k_2)$ we find:

$$L_{\mu\nu} = 2k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{2\mu} + g_{\mu\nu}q^2. \tag{83}$$

Neglecting the electron mass:

$$\boxed{L_{\mu\nu} = 2k_{1\mu}k_{2\nu} + 2k_{1\nu}k_{2\mu} - 2g_{\mu\nu}k_1 \cdot k_2.}$$

From this expression we see that the leptonic tensor which describes unpolarized electrons is symmetrical.

4.4. *The Rosenbluth formula*

Let us calculate explicitly the components for the hadronic tensor $W_{\mu\nu}$, in terms of FFs G_E and G_M. Recalling the property that $Tr\vec{\sigma} \cdot \mathbf{A} = 0$, for any vector $\mathbf{A}$, we see that all terms for the components $W_{\mu\nu}$ which contain the product $G_E G_M$ vanish: this means that the unpolarized cross section of $eN-$scattering does not contain this interference term. The non-zero components of $W_{\mu\nu}$ are determined only by G_E^2 and G_M^2:

$$W_{00} = 4M_p^2 G_E^2,$$

$$W_{xx} = -q^2 G_M^2,$$

$$W_{yy} = -q^2 G_M^2.$$

Substituting these expressions in Eq. (78), one can find for the matrix element squared:

$$\left(\frac{q^2}{e^2}\right)^2 \overline{|\mathcal{M}|^2} = L_{00}W_{00} + (L_{xx} + L_{yy})W_{xx} = L_{00}4M_p^2 G_E^2 + (L_{xx} + L_{yy})(-q^2)G_M^2. \tag{84}$$

The necessary components of the leptonic tensor $L_{\mu\nu}$, calculated in the Breit system, are:

$$L_{00} = 4\epsilon_{1B}^2 + q^2 = -q^2 \cot^2 \frac{\theta_B}{2},$$

$$L_{yy} = -q^2,$$

$$L_{xx} = 4k_{1x}^2 - q^2 = -q^2 \left(1 + \cot^2 \frac{\theta_B}{2}\right).$$

Substituting the corresponding terms in Eq. (84) we have:

$$\overline{|\mathcal{M}|^2} = \left(\frac{e^2}{q^2}\right)^2 \left[-q^2 \cot^2 \frac{\theta_B}{2} 4M_p^2 G_E^2 + (-2q^2 - q^2 \cot^2 \frac{\theta_B}{2})(-q^2 G_M^2)\right],$$

which becomes in the Lab system:

$$\overline{|\mathcal{M}|^2} = \left(\frac{e^2}{q^2}\right)^2 4M_p^2(-q^2)\left[2\tau G_M^2 + \frac{\cot^2\frac{\theta_e}{2}}{1+\tau}(G_E^2+\tau G_M^2)\right]. \tag{85}$$

We can then find the following formula for the cross section, $d\sigma/d\Omega_e$, in the Lab system, in terms of the electromagnetic FFs G_E and G_M (Rosenbluth formula[4]) ‡:

$$\frac{d\sigma}{d\Omega_e} = \frac{\alpha^2}{-q^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\left[2\tau G_M^2 + \frac{\cot^2\frac{\theta_e}{2}}{1+\tau}\left(G_E^2+\tau G_M^2\right)\right], \tag{86}$$

where $\alpha = e^2/4\pi \simeq 1/137$ is the fine structure constant.

Taking into account Eq. (65) and the following relation between the energy ϵ_2 and the angle θ_e of the scattered electron:

$$\epsilon_2 = \frac{\epsilon_1}{1+2\dfrac{\epsilon_1}{M_p}\sin^2\dfrac{\theta_e}{2}}, \tag{87}$$

the differential cross section can be written in the following form:

$$\frac{d\sigma}{d\Omega_e} = \sigma_M\left[2\tau G_M^2\tan^2\frac{\theta_e}{2} + \frac{G_E^2+\tau G_M^2}{1+\tau}\right], \tag{88}$$

with

$$\sigma_M = \frac{\alpha^2}{-q^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\frac{\cos^2\dfrac{\theta_e}{2}}{\sin^2\dfrac{\theta_e}{2}} = \left(\frac{\alpha}{2\epsilon_1}\right)^2\frac{\cos^2\dfrac{\theta_e}{2}}{\sin^4\dfrac{\theta_e}{2}}\frac{1}{\left(1+2\dfrac{\epsilon_1}{M_p}\sin^2\dfrac{\theta_e}{2}\right)},$$

where σ_M is the Mott cross section, for the scattering of unpolarized electrons by a point charge particle (with spin 1/2).

Note that the very specific $\cot^2\dfrac{\theta_e}{2}$-dependence of the cross section for eN-scattering results from the assumption of one-photon mechanism for the considered reaction. This can be easily proved,[31] by crossing symmetry considerations, looking to the annihilation channel, $e^+ + e^- \to \overline{p} + p$. In the CMS of such reaction, the one-photon mechanism induces a simple and evident $\cos^2\theta$-dependence of the corresponding differential cross section, due to the C-invariance of the hadron electromagnetic interaction, and unit value of the photon spin.

The particular $\cot^2\dfrac{\theta_e}{2}$-dependence of the differential eN-cross section is at the basis of the method to determine both nucleon electromagnetic FFs, G_E and G_M, using the linearity of the *reduced* cross section:

$$\sigma_{red} = \frac{\dfrac{d\sigma}{d\Omega_e}}{\dfrac{\alpha^2}{-q^2}\left(\dfrac{\epsilon_2}{\epsilon_1}\right)^2},$$

as a function of $\cot^2\frac{\theta_e}{2}$ (Rosenbluth fit or Rosenbluth separation). One can see that the backward eN-scattering ($\theta_e = \pi, \cot^2\frac{\theta_e}{2} = 0$) is determined by the magnetic FF only, and that the slope for σ_{red} is sensitive to G_E^2 (Fig. 4).

At large q^2, for $\tau \gg 1$, the differential cross-section $d\sigma/d\Omega_e$ (with unpolarized particles) is insensitive to G_E: the corresponding combination of the nucleon FFs, $G_E^2 + \tau G_M^2$ is dominated by the G_M contribution, due to the following reasons:

- $G_M/G_E \simeq \mu_p$, where μ_p is the proton magnetic moment, so $G_M^2/G_E^2 \simeq 2.79^2 \simeq 8$;
- The factor τ increases the G_M^2 contribution at large momentum transfer, where $\tau \gg 1$.

Therefore ep−scattering (with unpolarized particles) is dominated by the magnetic term, at large values of momentum transfer. The same holds for en−scattering, even at relatively small values of q^2, due to the smaller values of the neutron electric FF.

As a result, for the exact determination of the proton electric FF, in the region of large momentum transfer, and for the neutron electric FF - at any value of q^2, polarization measurements are required and in particular those polarization observables which are determined by the product $G_E G_M$, and are, therefore, more sensitive to G_E.

There are at least two different classes of polarization experiments of such type: the scattering of longitudinally polarized electrons by polarized target (with polarization in the reaction plane, but perpendicular to the direction of the three-momentum

‡More exactly, the original formula has been written in terms of the Dirac (F_1) and Pauli (F_2) form factors.

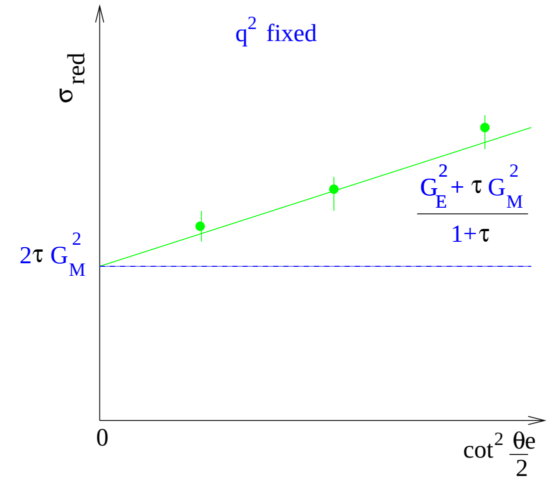

Fig. 4. Illustration of the Rosenbluth separation for the elastic differential cross section for eN-scattering.

transfer) $\vec{e}+\vec{p}\to e+p$, or the measurement of the ratio of transversal to longitudinal proton polarization (in the reaction plane) for the scattering of longitudinally polarized electrons by unpolarized target, $\vec{e}+p\to e+\vec{p}$.

In principle, there are some components of the depolarization tensor (characterizing the dependence of the final proton polarization on the target polarization (for the scattering of unpolarized electrons, $e+\vec{p}\to e+\vec{p}$) which are also proportional to G_EG_M, and therefore can be used for the determination of the nucleon electric FF.[1,2,44]

Both experiments (with polarized electron beam) have been realized: $p(\vec{e},\vec{p})e$ for the determination of the proton electric FF, G_{Ep}[21] and, for the determination of the neutron electric FF, G_{En}, $d(\vec{e},e'\vec{n})p$ and $d(\vec{e},e'n)\vec{p}$.[46]

4.5. *Polarization observables*

In general the hadronic tensor $W_{\mu\nu}$, for ep elastic scattering, contains four terms, related to the four possibilities of polarizing the initial and final protons:

$$W_{\mu\nu}=W_{\mu\nu}^{(0)}+W_{\mu\nu}(\vec{P}_1)+W_{\mu\nu}(\vec{P}_2)+W_{\mu\nu}(\vec{P}_1,\vec{P}_2),$$

where $\vec{P}_1$, $(\vec{P}_2)$ is the polarization vector of the initial (final) proton. The first term corresponds to the unpolarized case, the second (third) term corresponds to the case when the initial (final) proton is polarized, and the last term describes the reaction when both protons (initial and final) are polarized. The 2×2 density matrix for a nucleon with polarization $\vec{P}$ can be written as: $\rho=\frac{1}{2}\left(1+\vec{\sigma}\cdot\vec{P}\right)$.

Let us consider the case when only the final proton is polarized ($\vec{P}=\vec{P}_2$):

$$W_{\mu\nu}(\vec{P})=\frac{1}{2}TrF_\mu F_\nu^\dagger\vec{\sigma}\cdot\mathbf{P}.$$

For the scattering of longitudinally polarized electrons (by unpolarized target), only the x and z components of the polarization vector $\mathbf{P}$ do not vanish. To find these components, let us calculate the tensors $W_{\mu\nu}(P_x)$ and $W_{\mu\nu}(P_z)$.

$$W_{\mu\nu}(P_x)=\frac{1}{2}TrF_\mu F_\nu^\dagger\sigma_x.$$

Let us start [§] from the calculation of the components $F_\nu^\dagger$:

$$F_\nu^\dagger=\begin{cases}2M_pG_E & ,\nu=0,\\ -i\sqrt{-q^2}G_M\sigma_y & ,\nu=x,\\ i\sqrt{-q^2}G_M\sigma_x & ,\nu=y,\\ 0 & ,\nu=z.\end{cases}\qquad(89)$$

[§]We will take into account the fact that $G_E(q^2)$ and $G_M(q^2)$ are real functions of (q^2) in the space-like region.

Therefore, one can find easily (using $\sigma_x\sigma_y = i\sigma_z,\ \sigma_y\sigma_z = i\sigma_x, \sigma_z\sigma_x = i\sigma_y$):

$$F_\nu^\dagger \sigma_x = \begin{cases} 2M_pG_E\sigma_x & ,\nu = 0, \\ -\sqrt{-q^2}G_M\sigma_z & ,\nu = x, \\ i\sqrt{-q^2}G_M & ,\nu = y, \\ 0 & ,\nu = z. \end{cases} \tag{90}$$

This allows to write:

$$F_\mu F_\nu^\dagger \sigma_x = \begin{cases} 2M_pG_E & ,\mu = 0, \\ i\sqrt{-q^2}G_M\sigma_y & ,\mu = x, \\ -i\sqrt{-q^2}G_M\sigma_x & ,\mu = y, \\ 0 & ,\mu = z, \end{cases} \bigotimes \begin{cases} 2M_pG_E\sigma_x & ,\nu = 0, \\ -\sqrt{-q^2}G_M\sigma_z & ,\nu = x, \\ i\sqrt{-q^2}G_M & ,\nu = y, \\ 0 & ,\nu = z. \end{cases} \tag{91}$$

As we have to calculate the trace, recalling that $Tr\sigma_{x,y,z} = 0$, we can see that the non-zero components of the hadronic tensor $W_{\mu\nu}(P_x)$ are:

$$\begin{aligned} W_{0y}(P_x) &= i\sqrt{-q^2}\, 2M_pG_EG_M, \\ W_{y0}(P_x) &= -i\sqrt{-q^2}\, 2M_pG_EG_M. \end{aligned} \tag{92}$$

So we proved here that only two components of $W_{\mu\nu}(P_x)$ are different from zero: they are equal in absolute value and opposite in sign: it follows that $W_{\mu\nu}(P_x)$ is an antisymmetrical tensor. Therefore, the product $L_{\mu\nu}W_{\mu\nu}(P_x)$ vanishes: the product of a symmetrical tensor and an asymmetrical tensor is zero. This means that the polarization of the final proton vanishes, if the electron is unpolarized: **unpolarized electrons can not induce polarization of the scattered proton**. This is a property of the one-photon mechanism $for\ any\ elastic\ electron-hadron\ scattering$ and of the hermiticity of the Hamiltonian for the hadron electromagnetic interaction. Namely the hermiticity condition allows to prove that the hadron electromagnetic FFs are real functions of the momentum transfer squared in the space-like region. On the other hand, in the time-like region, which is scanned by the annihilation processes, $e^- + e^+ \leftrightarrow \overline{p} + p$, the nucleon electromagnetic FFs are complex functions of q^2, if $q^2 \geq 4m_\pi^2$, where m_π is the pion mass. This is due to the unitarity condition, which can be illustrated as in Fig. 5.

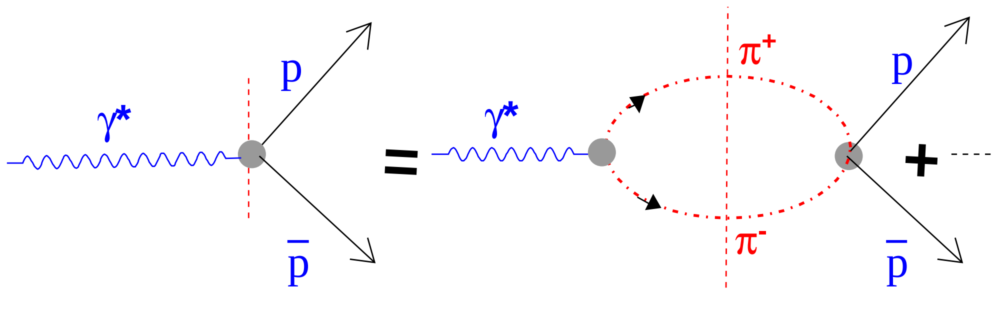

Fig. 5. The unitarity condition for proton electromagnetic FFs in the time-like region of momentum transfer squared. Vertical line on the right side crosses the pion lines, describing real particles (on mass shell). The dotted line denotes other possible multi-pion states, in the chain of the following transitions: $\gamma^* \to n\pi \to p\overline{p}$, where n is the number of pions in the intermediate state.

The complexity of nucleon FFs (in the time-like region) results in specific polarization phenomena, for the annihilation processes $e^+ + e^- \leftrightarrow \overline{p} + p$, which are different from the case of elastic ep–scattering. For example, the polarization of the final proton (or antiproton) is different from zero, even in the case of collisions of unpolarized leptons: this polarization is determined by the product $\mathcal{I}mG_EG_M^*$ (and, therefore, vanishes in the case of elastic ep-scattering, where FFs are real). Note that two-photon exchange in ep-elastic scattering is also generating complex amplitudes. So the interference between one and two-photon amplitudes induces nonzero proton polarization, but small in absolute value, as it is proportional to α.

Numerous experiments[47] have been done with the aim to detect such polarization at small momentum transfer $|q^2| \leq 1$ GeV^2, but with negative result, at a percent level. Only recently the above mentioned interference was experimentally detected, measuring the asymmetry in the scattering of transversally polarized electrons by an unpolarized proton target.[48,49]

Note that at very large momentum transfer, the relative role of two-photon amplitudes may be increased (violating the counting in α), due to the steep q^2-decreasing of hadronic electromagnetic FFs.

Note also that the analytical properties of the nucleon FFs, considered as functions of the complex variable $z = q^2$, result in a specific asymptotic behavior, as they obey to the Phragmèn-Lindelöf theorem:[50]

$$\lim_{q^2 \to -\infty} F^{(SL)}(q^2) = \lim_{q^2 \to \infty} F^{(TL)}(q^2). \tag{93}$$

The existing experimental data about the proton FFs in the time-like region up to 15 GeV^2, seem to contradict this theorem.[51] More exactly, one can prove that, if one FF, electric or magnetic; satisfies the relation (93), then the other one violates this theorem, i.e., the asymptotic condition does not apply.

Let us consider now the proton polarization in the z-direction:

$$W_{\mu\nu}(P_z) = \frac{1}{2} Tr F_\mu F_\nu^\dagger \sigma_z.$$

First, we calculate the components of $F_\nu^\dagger \sigma_z$:

$$F_\nu^\dagger \sigma_z = \begin{cases} 2M_p G_E \sigma_z & , \nu = 0, \\ \sqrt{-q^2} G_M \sigma_x & , \nu = x, \\ \sqrt{-q^2} G_M \sigma_y & , \nu = y, \\ 0 & , \nu = z. \end{cases} \tag{94}$$

Therefore we find:

$$F_\mu F_\nu^\dagger \sigma_z = \begin{cases} 2M_p G_E & , \mu = 0, \\ i\sqrt{-q^2} G_M \sigma_y & , \mu = x, \\ -i\sqrt{-q^2} G_M \sigma_x & , \mu = y, \\ 0 & , \mu = z, \end{cases} \bigotimes \begin{cases} 2M_p G_E \sigma_z & , \nu = 0, \\ \sqrt{-q^2} G_M \sigma_x & , \nu = x, \\ \sqrt{-q^2} G_M \sigma_y & , \nu = y, \\ 0 & , \nu = z. \end{cases} \tag{95}$$

We see that $W_{0\nu}(P_z) = W_{\nu 0}(P_z) = 0$, for any ν, and no interference term $G_E G_M$ is present. The nonzero components of $W_{\mu\nu}(P_z)$ are:

$$\begin{aligned} W_{xy}(P_z) &= -iq^2 G_M^2, \\ W_{yx}(P_z) &= iq^2 G_M^2, \end{aligned} \tag{96}$$

from where we see that $W_{\mu\nu}(P_z)$ is an antisymmetrical tensor, which depends on G_M^2 and that $P_x/P_z \propto G_E/G_M$.

4.5.1. *Polarized electron*

The leptonic tensor, $L_{\mu\nu}$, in case of unpolarized particles, contains only one term. For longitudinally polarized electrons, the polarization is characterized by the helicity λ, which takes values ± 1, corresponding to the direction of spin parallel or antiparallel to the electron three-momentum.

Relativistic description of the electron polarization

Using the expression 57 for the density matrix ρ, let us calculate the leptonic tensor $L_{\mu\nu}(\lambda)$, corresponding to the scattering of longitudinally polarized electrons (neglecting the electron mass):

$$L_{\mu\nu}(\lambda) = \frac{1}{2}Tr\gamma_\mu\hat{k}_1(1+\lambda\gamma_5)\gamma_\nu\hat{k}_2 = \frac{1}{2}Tr\gamma_\nu\hat{k}_1\gamma_\nu\hat{k}_2 + \frac{\lambda}{2}Tr\gamma_\nu\hat{k}_1\gamma_5\gamma_\nu\hat{k}_2 = L^{(0)}_{\mu\nu} + \lambda L^{(1)}_{\mu\nu}. \tag{97}$$

The tensor $L^{(0)}_{\mu\nu}$ corresponds to the scattering of unpolarized electrons:

$$L^{(0)}_{\mu\nu} = 2k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu} - g_{\mu\nu}k_1 \cdot k_2. \tag{98}$$

The tensor $L^{(1)}_{\mu\nu}$, describing the dependence on the longitudinal electron polarization can be written in the following form:

$$L^{(1)}_{\mu\nu} = \frac{1}{2}Tr\gamma_\mu\hat{k}_1\gamma_\nu\hat{k}_2\gamma_5 = -\frac{1}{2}Tr\gamma_\mu\gamma_\nu\hat{k}_1\hat{k}_2\gamma_5 = 2i\epsilon_{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}. \tag{99}$$

We applied another property of γ_5, that is:

$$Tr\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 = -4i\epsilon_{\mu\nu\rho\sigma}.$$

Taking into account the conservation of four-momentum in the electron vertex: $k_1 = k_2 + q$, we can rewrite the tensor $L^{(1)}_{\mu\nu}$ in the following form, which is more convenient in this frame:

$$L^{(1)}_{\mu\nu} = 2i\epsilon_{\mu\nu\rho\sigma}q_\rho k_{1\sigma}. \tag{100}$$

The three-vector **q** has only nonzero $z-$component, in the Breit system. The tensor $\epsilon_{\mu\nu\rho\sigma}$ is defined in such way that $\epsilon_{xyz0} = +1$.

The general expression for the leptonic tensor in case of longitudinally polarized electrons is:

$$L_{\mu\nu} = L^{(0)}_{\mu\nu} + L_{\mu\nu}(\lambda_1) + L_{\mu\nu}(\lambda_2) + L_{\mu\nu}(\lambda_1, \lambda_2), \tag{101}$$

where the first term, considered previously, describes the collision where the initial and final electrons are unpolarized, the second (third) term describes the case when the initial (final) electron is longitudinally polarized, and the last terms holds when both electrons are longitudinally polarized.

If only the initial electron is polarized, $\lambda_1 = \lambda$, one can write for $L_{\mu\nu}$:

$$L_{\mu\nu}(\lambda) = 2i\lambda\epsilon_{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}. \tag{102}$$

The effect of the electron polarization is described by an antisymmetrical tensor $L_{\mu\nu}(\lambda)$. If the initial proton is unpolarized, again, being described by symmetrical tensor, the total result will be zero. This result holds because FFs are real, so it does not apply to the time-like region.

Let us consider the x and z components.

x-component

Let us consider the product of the leptonic $L_{\mu\nu}(\lambda)$ and hadronic $W_{\mu\nu}(P_x)$ tensors, for the x component of the final proton polarization:

$$L_{\mu\nu}(\lambda)W_{\mu\nu}(P_x) = L_{0y}(\lambda)W_{0y}(P_x) + L_{y0}(\lambda)W_{y0}(P_x) = L_{0y}(\lambda)\left[W_{0y}(P_x) - W_{y0}(P_x)\right] = 2L_{0y}(\lambda)W_{0y}(P_x). \quad (103)$$

Taking into account that: $L_{0y} = 2i\lambda\epsilon_{0y\alpha\beta}k_{1\alpha}k_{2\beta}$ the only non-zero terms correspond to $\alpha = x$ and $\beta = z$ or $\alpha = z$ and $\beta = x$. Therefore:

$$L_{0y}(\lambda) = 2i\lambda\left(\epsilon_{0yxz}k_{1x}k_{2z} + \epsilon_{0yzx}k_{1z}k_{2x}\right) = 2i\lambda\epsilon_{0yxz}(k_{1x}k_{2z} - k_{1z}k_{2x}) = i\lambda q^2 \cot\frac{\theta_B}{2},$$

with $\epsilon_{0yxz} = 1$, and using Eqs. (61) and (62).

We finally find:

$$L_{\mu\nu}(\lambda)W_{\mu\nu}(P_x) = -4\lambda M_p q^2\sqrt{-q^2}\cot\frac{\theta_B}{2}G_E G_M. \quad (104)$$

z-component

Similarly, considering the antisymmetry of both tensors $L_{\mu\nu}(\lambda)$ and $W_{\mu\nu}(P_z)$, one can find:

$$L_{\mu\nu}(\lambda)W_{\mu\nu}(P_z) = 2i\lambda\epsilon_{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}W_{\mu\nu}(P_z) = 4\epsilon_{xy0z}W_{xy}(P_z)\left(\epsilon_{1B}k^z_{2B} - \epsilon_{2B}k^z_{1B}\right) = 4\lambda q^2\frac{G_M^2}{\sin\theta_B/2}. \quad (105)$$

4.6. *Final formulas*

The polarization **P** of the scattered proton can be written as:

$$\mathbf{P}\frac{d\sigma}{d\Omega_e} = \frac{\alpha^2}{4\pi^2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^2\frac{L_{\mu\nu}}{M_p^2}\vec{P}_{\mu\nu},$$

with $\vec{P}_{\mu\nu} = \frac{1}{2}(\mathcal{T}r\mathcal{F}_\mu\mathcal{F}_\nu^\dagger\vec{\sigma})$, so that $P^{(z)}_{\mu\nu} = W_{\mu\nu}(P_z)$ and $P^{(x)}_{\mu\nu} = W_{\mu\nu}(P_x)$.

Using Eq. (66) one can find the following expressions for the components P_x and P_z of the proton polarization vector (in the scattering plane) - in terms of the proton electromagnetic FFs:

$$DP_x = -2\lambda\cot\frac{\theta_e}{2}\sqrt{\frac{\tau}{1+\tau}}G_E G_M, \quad DP_z = \lambda\frac{\epsilon_1+\epsilon_2}{M_p}\sqrt{\frac{\tau}{1+\tau}}G_M^2, \quad (106)$$

where D is proportional to the differential cross section with unpolarized particles:

$$D = 2\tau G_M^2 + \cot^2\frac{\theta_e}{2}\frac{G_E^2+\tau G_M^2}{1+\tau}. \quad (107)$$

So, for the ratio of these components one can find the following formula:

$$\frac{P_x}{P_z} = \frac{P_t}{P_\ell} = -2\cot\frac{\theta_e}{2}\frac{M_p}{\epsilon_1+\epsilon_2}\frac{G_E(q^2)}{G_M(q^2)} \quad (108)$$

which clearly shows that a measurement of the ratio of transverse and longitudinal polarization of the recoil proton gives a direct measurement of the ratio of electric and magnetic FFs, $G_E(q^2)/G_M(q^2)$.

In the same way it is possible to calculate the dependence of the differential cross section for the elastic scattering of the longitudinally polarized electrons by a **polarized** proton target, with polarization $\mathcal{P}$, in the above defined coordinate system:

$$\frac{d\sigma}{d\Omega_e}(\mathcal{P}) = \left(\frac{d\sigma}{d\Omega_e}\right)_0(1 + \lambda\mathcal{P}_x A_x + \lambda\mathcal{P}_z A_z), \quad (109)$$

where the asymmetries A_x and A_z (or the corresponding analyzing powers) are related in a simple and direct way, to the components of the final proton polarization:

$$A_x = P_x, \quad A_z = -P_z. \quad (110)$$

This holds in the framework of the one-photon mechanism for elastic $ep-$scattering. Note that the quantities A_x and P_x have the same sign and absolute value, but the components A_z and P_z, being equal in absolute value, have opposite sign.

These two different polarization experiments in elastic electron-proton scattering, namely the scattering with longitudinally polarized electrons by a polarized proton target (with polarization in the reaction plane) from one side and the measurement of the components of the final proton polarization (again in the reaction plane) in the scattering of longitudinally polarized electrons by an unpolarized proton target, from another side, bring the same physical information, concerning the electromagnetic FFs of proton.

Note that the P_y-component of the proton polarization vanishes in the scattering of polarized and unpolarized electrons, as well. This results from the one-photon mechanism and the fact that G_E and G_M are real. For the same reasons, the corresponding analyzing power, A_y, also vanishes.

4.7. *Discussion*

The expressions of the unpolarized cross section and of the polarization observables in terms of FFs given above for elastic ep-scattering, hold in the framework of the one-photon mechanism.

There are at least two different sources of corrections to these relations:

- the standard radiative corrections;
- the electroweak corrections.

These last corrections arise from the interference of amplitudes, corresponding to the exchange of γ and $Z-$boson. The relative value of these contributions is characterized by the following dimensionless parameter:

$$G_{eff} = \frac{G_F}{2\sqrt{2}\alpha\pi}|q^2| \simeq 10^{-4}\frac{|q^2|}{\mathrm{GeV}^2},$$

where G_F is the standard Fermi constant of the weak interaction, $G_F \simeq 10^{-5}/M_p^2$.

So, for $|q^2| \leq 10$ GeV^2, the electroweak corrections are negligible, for the polarization phenomena considered above. However, note that the $\gamma\otimes Z$-interference is not only inducing (small) corrections to the results of the one-photon considerations, but it induces also a new class of polarization observables of P-odd nature, i.e., with violation of the P-invariance. The simplest of them is the P-odd asymmetry of the scattering of longitudinally polarized electrons by an unpolarized proton target $\vec{e} + p \to e + p$ (the detection of the polarization of the scattered particles is not required). As this asymmetry vanishes in the one-photon mechanism, it is proportional to G_{eff}, at relatively small momentum transfer squared.

Let us turn to the QED radiative corrections. They appear essentially in the differential cross section, and they have been discussed, for example, in[36] and more recently by.[38]

For polarization phenomena, it can be proved[38] that, in case of soft photons, the contribution of radiative corrections can be explicitly factorized. Therefore, this contribution, which is important for the differential cross section, cancels in polarization effects. Radiation of non-soft photons by electrons (in initial and final states) results in corrections, which are different for the components P_x and P_z. Such corrections can be calculated in a model independent way, in the framework of the standard QED, inducing effects of a few percent.[39]

Model dependent radiative corrections can not be uniquely calculated. This concerns, first of all, the virtual Compton scattering on nucleons, which is driven by the amplitude of the process $\gamma^* + p \to \gamma + p$, with very complicated spin structure and with different mechanisms, as, for example, pion exchange in t-channel and Δ-exchange in s-channel. These contributions can be estimated to give corrections of 1-3 %.

The most intriguing part of the radiative corrections is due to the two-photon exchange at large momentum transfer, with comparable virtuality of the two photons. Polarization phenomena for elastic positron scattering and for elastic scattering of positive and negative muons are the same as in case of electron scattering, only in case of one photon exchange.

Radiative corrections modify not only the absolute value, but also the dependence of the observables on the relevant kinematical variables and, in case of unpolarized cross section, at large momentum transfer they can reach 30-40%.[52] Therefore, it appears necessary to introduce high order corrections,[41] what can be done in frame of the lepton structure functions (LSF) method.[37,53]

This formalism equally applies to en-elastic scattering, in the case of free neutron. As typically a target like d or ^3He is used, specific considerations apply, which are outside the present notes (see Ref.[54]). The present formalism is valid in case of elastic $e+^3He$ and $e+^3H$ scattering, and, in general, for elastic scattering of electrons on any spin 1/2 target.

5. Symmetries and two photon exchange

The discrepancy of recent experimental results on *ed* and *ep* elastic scattering obtained in different experimental set-ups and/or with different methods, lead to the suggestion that, beyond possible systematic effects not taken properly into account, they could result from the presence of a different reaction mechanism, the exchange of two photons.[31] This is not a new idea: in the 70's much theoretical and experimental work was devoted to this problem. More than 25 years ago it was observed[27–30] that the simple rule of α-counting for the estimation of the relative role of two-photon contribution to the amplitude of elastic *ed*–scattering, does not hold at large momentum transfer. Using a Glauber approach for the calculation of multiple scattering contributions,[26] it was shown that the relative role of two-photon exchange can be essentially increased in the region of high momentum transfer. It was also shown that this effect can be observed in particular in *ed*-elastic scattering , due to the steep decreasing of the deuteron form factors. Moreover the relative role of two-photon contributions has to be even larger for heavy nuclei (like ^3He or ^4He) in comparison with deuteron. This effect would then manifest at relatively small momentum transfer - of the order of 1 GeV^2 - especially in the region of diffractive minima. The argument for the possible increase of the relative role of two-photon exchange at large momentum transfer follows from the fact that this momentum has to be shared between the two photons, which results in a non negligible two-photon amplitude. However, in[27] the two-photon amplitude is purely imaginary, at least at very small scattering angles, so it cannot interfere with the one-photon exchange amplitude. The experiments in the 70's were mainly focused on the difference between electron and positron elastic scattering on the proton (for a review, see[55]). The precision of the data does not allow to see the evidence of an effect lower than a few percent. Note that this is also the size of those radiative corrections which contain odd terms. Presently, many efforts are devoted to precise measurements of the difference between $e^\pm p$ elastic scattering at Novosibirsk, JLab and DESY.

One should also note that no experimental evidence of 2γ echange (more exactly, of the real part of the $1\gamma \bigotimes 2\gamma$ interference) has been found in the experimental data, searching for non linearities in the Rosenbluth plots for electron elastic scattering on particles with spin zero,[56] one half,[57] and one.[31] An analysis of asymmetry in the angular distributions for the BABAR data[9] also does not show evidence of two photon contribution, in the limit of the uncertainty of the data.[58]

Let us stress that the main advantage of the search of 2γ in TL region is that the information is fully contained in the angular distribution (which is equivalent to the charge asymmetry). In the same measurement, the odd terms corresponding to two photon exchange can be singled out (whereas in SL region, in case of two photon exchange it is necessary to measure electron and positron scattering, in the same kinematical conditions). Two photon exchange effects cancel if one does not measure the charge of the outgoing lepton, or in the sum of the cross section at complementary angles, allowing to extract the moduli of the true FFs.[33]

5.1. *Helicity amplitudes for binary reactions with spins* $1/2 + 1/2 \to 1/2 + 1/2$

N	1/2	1/2	→	1/2	1/2	N	1/2	1/2	→	1/2	1/2
1)	+	+	→	+	+	9)	-	+	→	+	+
2)	+	+	→	+	-	10)	-	+	→	+	-
3)	+	+	→	-	+	11)	-	+	→	-	+
4)	+	+	→	-	-	12)	-	+	→	-	-
5)	+	-	→	+	+	13)	-	-	→	+	+
6)	+	-	→	+	-	14)	-	-	→	+	-
7)	+	-	→	-	+	15)	-	-	→	-	+
8)	+	-	→	-	-	16)	-	-	→	-	-

The total number of amplitudes for a binary reaction is $(2S_1 + 1)(2S_2 + 1)(2S_3 + 1)(2S_4 + 1)$, where S_i, $i = 1 - 4$, is the spin of the *i*- particle involved, see Table 5. However, not all of them are independent, but they are related by symmetry properties:

- Parity conservation: it implies the identity of the amplitudes obtained when reversing all spins: it reduces the number of amplitudes from $16 \to 8$.
- Identity of initial and final states: it gives two more conditions: 2=5, 3=9=8 (9 was already equal to 8).

We are left with 16/2-2=6 amplitudes. In Table 6, they are classified with, in the right column, the ones which require a spin-flip of the projectile.

N	e	p	→	e	p	N	e	p	→	e	p
1)	+	+	→	+	+	4)	+	+	→	-	+
2)	+	+	→	+	-	5)	+	+	→	-	-
3)	+	-	→	+	-	6)	+	-	→	-	+

In case of high energy electrons (where $m_e/E\Lambda\overline{\Lambda}1$), helicity conservation strongly suppress the amplitudes 4-6. The amplitudes 1 and 3 correspond to $\Delta S = 0$, the amplitudes 2,4-6 correspond to $\Delta S = 1$. This require $L = 0$ and $L = 1$, respectively, in order to conserve parity.

This analysis is better done in the annihilation channel. For illustration, let us consider firstly the one-photon mechanism for $e^+ + e^- \to p + \overline{p}$. The conservation of the total angular momentum $\mathcal{J}$ allows only one value, $\mathcal{J} = 1$, and the quantum numbers of the photon.

The selection rules with respect to the C- and P invariances allow two states for e^+e^- (and $p\overline{p}$):

$$S = 1,\ \ell = 0 \text{ and } S = 1,\ \ell = 2 \text{ with } \mathcal{J}^P = 1^-, \tag{111}$$

where S is the total spin and ℓ is the orbital angular momentum of the $e^+ + e^-$ system. As a result the θ dependence of the cross section for $e^+ + e^- \to \overline{p} + p$, in the one-photon exchange mechanism must have the following general form:

$$\frac{d\sigma}{d\Omega}(e^+ + e^- \to \overline{p} + p) \simeq a(t) + b(t)\cos^2\theta, \tag{112}$$

where $a(t)$ and $b(t)$ are definite quadratic contributions of $G_E(t)$ and $G_M(t)$, $a(t)$ and $b(t) \geq 0$ at $t \geq 4M_p^2$.

Using the kinematical relation (see below):

$$\cos^2\theta = \frac{1+\epsilon}{1-\epsilon} = \frac{\cot^2\theta_e/2}{1+\tau} + 1 \tag{113}$$

between the variables in the CMS of $e^+ + e^- \to \overline{p} + p$ and in the LAB system for $e^- + p \to e^- + p$, it appears clearly that the one-photon mechanism generates a linear ϵ dependence (or $\cot^2\theta_e/2$) of the Rosenbluth differential cross section for elastic ep-scattering in Lab system.

Similarly, let us consider the $\cos\theta$ dependence of the $1\gamma \bigotimes 2\gamma$-interference contribution to the differential cross section of $e^+ + e^- \to \overline{p} + p$. The spin and parity of the 2γ-states is not fixed, in general, but only a positive value of C-parity, $C(2\gamma) = +1$, is allowed. An infinite number of states with different quantum numbers (for $e^+ + e^-$ and $\overline{p} + p$) can contribute, and their relative role is determined by the dynamics of the process $\gamma^* + \gamma^* \to \overline{p} + p$, with both virtual photons.

But the $\cos\theta$ dependence of the $1\gamma\otimes 2\gamma$ interference contribution to the differential cross section can be predicted on the basis of its C-odd nature:

$$\frac{d\sigma^{(int)}}{d\Omega}(e^+ + e^- \to \overline{p} + p) = \cos\theta[c_0(t) + c_1(t)\cos^2\theta + c_2(t)\cos^4\theta + ...], \tag{114}$$

where $c_i(t)$, $i = 0, 1..$ are real coefficients, which are functions of t, only. This odd $\cos\theta$ dependence is essentially different from the even $\cos\theta$ dependence of the cross section for the one-photon approximation.

5.1.1. *Kinematical relation between Lab electron-scattering angle in $e + p \to e + p$ and CMS antiproton angle in $\overline{p} + p \to e^+ + e^-$*

Let us prove the following relation

$$\cos^2\theta = \frac{1+\epsilon}{1-\epsilon} = \frac{\cot^2\theta_e/2}{1+\tau} + 1, \tag{115}$$

where θ_e is the laboratory scattering angle of the electron in elastic ep scattering and θ is the CMS angle of the antiproton produced in the annihilation: $e^- + e^+ \to \overline{p} + p$ with respect to the beam direction.

This kinematical relation shows clearly the physical link between the linear ϵ dependence of the Rosenbluth differential cross section for elastic ep-scattering in Lab system (or $\cot^2\theta_e/2$) and the even distribution in $\cos^2\theta$ for the differential annihilation cross section in $\overline{p} + p \leftrightarrow e^+ + e^-$.

Crossing symmetry allows to connect scattering and annihilation channels (change a particle into antiparticle, change sign to the momenta):

$$e^-(k_1) + p(p_1) \to e^-(k_2) + p(p_2),\ e^-(k_1) + e^+(-k_2) \to \overline{p}(-p_1) + p(p_2).$$

(1) Let us calculate s and t in the scattering channel:

$$s=(p_1+k_1)^2=M_p^2+2\epsilon_1 M_p=M_p(M_p+2\epsilon_1)\to\epsilon_1=\frac{s-M_p^2}{2M_p}; \tag{116}$$

$$t=(k_1-k_2)^2=k_1^2+k_2^2-2\epsilon_1\epsilon_2+2|\mathbf{k}_1||\mathbf{k}_2|\cos\theta_e=-4\epsilon_1\epsilon_2\sin^2\frac{\theta_e}{2}. \tag{117}$$

where we assumed $m_e=0$ and we calculate t as function of the electron variables.

(2) The energy and momentum conservation are: $\epsilon_1+M_p=\epsilon_2+E_2;\ \mathbf{k}_1=\vec{k}_2+\mathbf{p}_2$;

(3) Let us express t from the hadron variables:

$$t=(p_2-p_1)^2=2M_p^2-2M_pE_2=2M_p^2-2M_p(\epsilon_1+M_p-\epsilon_2)=2M_p(\epsilon_2-\epsilon_1). \tag{118}$$

From the equality of Eqs. (117) and (118):

$$t=2M_p(\epsilon_2-\epsilon_1)=-4\epsilon_1\epsilon_2\sin^2\frac{\theta}{2}. \tag{119}$$

Hence

$$\epsilon_2=\frac{\epsilon_1}{1+2\dfrac{\epsilon_1}{M_p}\sin^2\dfrac{\theta}{2}}=\frac{M_p(s-M_p^2)}{2\left[M_p^2+(s-M_p^2)\sin^2\dfrac{\theta}{2}\right]}. \tag{120}$$

(4) Inserting the expression of ϵ_1 and ϵ_2 as a functions of s in Eq. 118:

$$\frac{1}{t}=-\frac{M_p^2}{(s-M_p^2)^2\sin^2\dfrac{\theta}{2}}-\frac{1}{s-M_p^2}. \tag{121}$$

(5) In the annihilation channel (CMS) one has $\tilde\epsilon_1=\tilde\epsilon_2=\tilde E_1=\tilde E_2=\epsilon$; $\mathbf{k}_1=-\mathbf{k}_2=\mathbf{k}$, $\mathbf{p}_1=-\mathbf{p}_2=\mathbf{p}\neq\mathbf{k}$:

$$s=(k_1-p_2)^2=M_p^2-2\tilde\epsilon_1^2+2\tilde\epsilon_2\vec{p}_2\cos\tilde\theta \tag{122}$$

$$t=(k_1+k_2)^2=2\tilde\epsilon_1^2-2\tilde\epsilon_1\tilde\epsilon_2\cos\widehat{k_1k_2}=4\tilde\epsilon_1^2, \tag{123}$$

from where we find the expression of $\cos\tilde\theta$ as a function of the invariants s and t:

$$\cos\tilde\theta=\frac{s-M_p^2+2\tilde\epsilon^2}{2\tilde\epsilon\sqrt{\tilde\epsilon^2-M_p^2}}\ \to\ \cos^2\tilde\theta=\frac{(s-M_p^2)^2+ts}{t\left(\dfrac{t}{4}-M_p^2\right)}+1. \tag{124}$$

Reminding that $\tau=-t/(4M_p^2)$, one finds

$$t\left(\frac{t}{4}-M_p^2\right)=-M_p^2t(\tau+1). \tag{125}$$

Inserting the relation $\left(\sin^2\dfrac{\theta}{2}\right)^{-1}=\cot^2\dfrac{\theta}{2}+1$ in Eq. (121), one finds

$$\cot^2\frac{\theta}{2}=\frac{(s-M_p^2)^2+ts}{-M_p^2t}. \tag{126}$$

(6) Comparing Eqs. (126) and (124) with the help of (125) one verifies the relation Eq. (115).

5.2. *Two photon exchange for ep scattering*

The exact calculation of the 2γ-contribution to the amplitude of the $e^\pm p\to e^\pm p$-process requires the knowledge of the matrix element for the double virtual Compton scattering, $\gamma^*+p\to\gamma^*+p$, in a large kinematical region of colliding energy and virtuality of both photons, and can not be done in a model independent form. However general properties of the hadron electromagnetic interaction, as C-invariance and crossing symmetry, give rigorous prescriptions for different observables for the elastic scattering of electrons and positrons by nucleons, in particular for the differential cross section and for the proton polarization, induced by polarized electrons. These concrete prescriptions help in identifying a possible manifestation of the two-photon exchange mechanism. For example, assuming a linear ϵ dependence of the elastic cross section in presence of 2γ-corrections is in contradiction with the C-invariance of the electromagnetic interaction (ϵ is the degree of polarization of the virtual photon).

If one takes into account the two-photon mechanism, the expressions of the matrix element and of the differential cross section, are essentially modified.

It is required, first of all, a generalization of the spin structure of the matrix element, which can be done, in analogy with elastic np-scattering,[59] using the general properties of the electron-hadron interaction, such as P-invariance and relativistic invariance.

Taking into account the identity of the initial and final states and the T-invariance of the electromagnetic interaction, we showed above that the processes $e^{\pm}N \to e^{\pm}N$, in which four particles with spin 1/2 participate, are characterized by six independent products of four-spinors, describing the initial and final fermions. The corresponding (model independent) parametrization of the matrix element can be done in many different but equivalent forms, in terms of six invariant complex amplitudes, $\mathcal{A}_i(s,Q^2)$, $i = 1-6$, which are functions of two independent variables, and $s=(k_1+p_1)^2$ is the square of the total energy of the colliding particles. In the physical region of the reaction $e^{\pm}N \to e^{\pm}N$ the conditions: $Q^2 \geq 0$ and $s \geq (M_p+m_e)^2 \simeq M_p^2$, apply.

Previously, another set of variables, ϵ and Q^2, which is equivalent to s and Q^2 (in Lab system) was considered. The variables ϵ and Q^2 are well adapted to the description of the properties of one-photon exchange for elastic eN-scattering, because, in this case, only the Q^2 dependence of the form factors has a dynamical origin, whereas the linear ϵ dependence in Eq. (88) is a trivial consequence of the one-photon mechanism. On the other hand, the variables s and Q^2 are better suited to the analysis of the implications from crossing symmetry.

The conservation of the lepton helicity, which is a general property of the electromagnetic interaction in electron-hadron scattering at high energy, reduces the number of invariant amplitudes for elastic eN-scattering, in general complex functions of s and Q^2, from six to three.

Therefore, we can write the following general parametrization of the spin structure of the matrix element for elastic eN-scattering, following the formalism of:[59]

$$\mathcal{M} = \frac{e^2}{Q^2}\overline{u}(k_2)\gamma_\mu u(k_1)\overline{u}(p_2)$$

$$\left[\mathcal{A}_1(s,Q^2)\gamma_\mu - \mathcal{A}_2(s,Q^2)\frac{\sigma_{\mu\nu}q_\nu}{2M_p} + \mathcal{A}_3(s,Q^2)\hat{K}\mathcal{P}_\mu\right]u(p_1), \tag{127}$$

$$K = \frac{k_1+k_2}{2},\ \mathcal{P} = \frac{p_1+p_2}{2},$$

where $\mathcal{A}_1 - \mathcal{A}_3$ are the corresponding invariant amplitudes.

In case of one-photon exchange these amplitudes are related to the nucleon form factors:

$$\mathcal{A}_1(s,Q^2) \to F_1(Q^2),\ \mathcal{A}_2(s,Q^2) \to F_2(Q^2),\ \mathcal{A}_3(s,Q^2) \to 0.$$

But in the general case (with multi-photon exchanges) the situation is more complicated, because:

- The amplitudes $\mathcal{A}_i(s,Q^2)$, $i = 1-3$, are complex functions of two independent variables, s and Q^2.
- The set of amplitudes $\mathcal{A}_i^{(-)}(s,Q^2)$ for the process $e^- + N \to e^- + N$ is different from the set $\mathcal{A}_i^{(+)}(s,Q^2)$ of corresponding amplitudes for positron scattering, $e^+ + N \to e^+ + N$, which means that the properties of positron scattering can not be derived from $\mathcal{A}_i^{(-)}(s,Q^2)$, as in case of the one-photon mechanism.
- The connection of the amplitudes $\mathcal{A}_i(s,Q^2)$ with the nucleon electromagnetic form factors, $F_{iN}(Q^2)$, is non-trivial, because these amplitudes depend on a large number of different quantities, as, for example, the form factors of the Δ-excitation - through the amplitudes of the virtual Compton scattering.

In this framework, the simple and transparent phenomenology of electron-hadron physics does not hold anymore, and in particular, it would be very difficult to extract information on the internal structure of a hadron in terms of electromagnetic form factors, which are real functions of one variable, from electron scattering experiments.

It has been proved that even in case of two-photon exchange, one can still use the formalism of form factors, taking into account the C-invariance of the electromagnetic interaction of hadrons.

The spin structure of the amplitudes $\mathcal{A}_1$ and $\mathcal{A}_2$ corresponds to exchange by vector particle (in t-channel), whereas the spin structure for the amplitude $\mathcal{A}_3$ corresponds to tensor exchange. Therefore, in case of $e^{\pm}N$–elastic scattering, in the $1\gamma + 2\gamma$ approximation, one can write the amplitudes $\mathcal{A}_{1,2}^{(\pm)}(s,Q^2)$ in the following form:

$$\mathcal{A}_{1,2}^{(\pm)}(s,Q^2) = \mp F_{1,2N}(Q^2) + \Delta\mathcal{A}_{1,2}^{(\pm)}(s,Q^2),$$

$$\Delta\mathcal{A}_{1,2}^{(+)}(s,Q^2) = \Delta\mathcal{A}_{1,2}^{(-)}(s,Q^2) \equiv \Delta\mathcal{A}_{1,2}(s,Q^2),$$

$$\mathcal{A}_3^{(+)}(s,Q^2) = \mathcal{A}_3^{(-)}(s,Q^2) \equiv \mathcal{A}_3(s,Q^2),$$

where the superscript $(\pm)$ corresponds to $e^{(\pm)}$ scattering. The amplitudes $\Delta\mathcal{A}_{1,2}(s,Q^2)$ and $\mathcal{A}_3(s,Q^2)$ contain only the 2γ-contribution, and are equal for $e^{(\pm)}$ scattering; $\Delta\mathcal{A}_{1,2}$ and $\mathcal{A}_3$ are of the order of α, $\alpha = e^2/(4\pi) = 1/137$.

Note that the difference in the spin structure of these amplitudes, Eq. (127), results in specific symmetry properties with respect to the change $x \to -x$ $\left(x = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\right)$:

$$\Delta\mathcal{A}_{1,2}(s,-x) = -\Delta\mathcal{A}_{1,2}(s,x), \mathcal{A}_3(s,-x) = +\mathcal{A}_3(s,x). \quad (128)$$

The x–odd behavior of $\Delta\mathcal{A}_{1,2}(s,x)$–contributions, corresponding to 2γ-exchange with $C = +1$, results from the C-odd character of the two vector-like spin structures, γ_μ and $\sigma_{\mu\nu}q_\nu$.

To prove this, let us consider, in addition to C-invariance, crossing symmetry, which allows to connect the matrix elements for the cross-channels: $e^- + p \to e^- + p$, s–channel, and $e^+ + e^- \to \bar{p} + p$, t–channel. The transformation from s- to t-channel can be realized by the following substitution:

$$k_2 \to -k_2,\ p_1 \to -p_1,$$

and for the invariant variables:

$$s = (k_1 + p_1)^2 \to (k_1 - p_1)^2,\ Q^2 = -(k_1 - k_2)^2 \to -(k_1 + k_2)^2 = -t.$$

The crossing symmetry states that the same amplitudes $\mathcal{A}_i(s,Q^2)$ describe the two channels, when the variables s and Q^2 scan the physical region of the corresponding channels. So, if $t \geq 4M_p^2$ and $-1 \leq \cos\theta \leq 1$ (θ is the angle of the proton production with respect to the electron three-momentum, in the center of mass (CMS) for $e^+ + e^- \to \bar{p} + p$), the amplitudes $\mathcal{A}_i(t,\cos\theta)$, $i = 1-3$, describe the process $e^+ + e^- \to \bar{p} + p$.

From C-invariance it follows that:

$$\mathcal{A}_3(t,-\cos\theta) = \mathcal{A}_3(t,+\cos\theta), \Delta A_{1,2}(t,-\cos\theta) = -\Delta A_{1,2}(t,+\cos\theta), \quad (129)$$

which is equivalent to the symmetry relations (128).

Therefore, it is incorrect to approximate the $1\gamma \bigotimes 2\gamma$ interference contribution to the differential cross section, Eq. (114) by a linear function in $\cos^2\theta$ (what may be found in recent literature), because it is in contradiction with the C-invariance of hadronic electromagnetic interaction.

6. The annihilation channel $\bar{p} + p \to e^+ + e^-$

The measurement of the differential cross section for the process $\bar{p} + p \to \ell^+ + \ell^-$ at a fixed value of the total energy s, and for two different angles θ, allows the separation of the two FFs, $|G_M|^2$ and $|G_E|^2$, and is equivalent to the Rosenbluth separation for the elastic ep-scattering. In TL region, this procedure is simpler, as it requires to change only one kinematical variable, $\cos\theta$, whereas, in SL region it is necessary to change simultaneously two kinematical variables: the energy of the initial electron and the electron scattering angle, fixing the momentum transfer squared, Q^2. Due to the limited statistics, the individual determination of the $|G_E|^2$ and $|G_M|^2$ contributions has not yet been realized in TL region.

In the TL region, the determination of a generalized FF requires to integrate the differential cross section over a wide angular range. One typically assumes that the G_E contribution plays a minor role in the cross section at large q^2 and the experimental results are usually given in terms of $|G_M|$, under the hypothesis that $G_E = 0$ or $|G_E| = |G_M|$. The first hypothesis is an arbitrary one. The second hypothesis is strictly valid at threshold only, i.e., for $\tau = q^2/(4M_p^2) = 1$, but there is no theoretical argument which justifies its validity at any other momentum transfer, where $q^2 \neq 4M_N^2$ (M_N is the nucleon mass, $N = p(n)$ for proton(neutron)). The $|G_M|$ values depend, in principle, on the kinematics where the measurement was performed and the angular range of integration. However, it turns out that these two assumptions for G_E lead to comparable values for $|G_M|$.

In annihilation channel, it is more convenient to perform the calculatations in CMS.

6.1. *Observables for $\bar{p} + p \to e^+ + e^-$*

The derivation given below is simplified by the use of 2×2 Pauli matrix, and 2-rank spinors, instead of 4×4 Dirac matrices and 4-rank spinors. It is a rigorous and simple derivation. The full derivation in the Dirac formalism can be found in Ref.[60]

Let us consider the annihilation reaction

$$\bar{p}(p_1) + p(p_2) \to e^-(k_1) + e^+(k_2) \quad (130)$$

in the CMS system, where an antiproton with three-momentum $\mathbf{p_1} = \mathbf{p}$ annihilates with a proton with three-momentum $\mathbf{p}_2 = -\mathbf{p}$. The transferred momentum is $t = s = (k_1 + k_2)^2 = 4E^2$ and (assuming $m_e = 0$) one has $\mathbf{k} = \mathbf{k_1} = -\mathbf{k_2}$; $E = |\mathbf{k}|$. We choose a reference system with the z axis along the beam momentum, and xz is the scattering plane. In this system the unit vectors are: $\mathbf{p} = (0, 0, 1)$ and $\mathbf{k} = (\sin\theta, 0, \cos\theta)$, with $\mathbf{p}\cdot\mathbf{k} = \cos\theta$.

The following relation holds (neglecting the electron mass):

$$\frac{\vec{\sigma}\cdot\mathbf{k}}{E+m_e} = \frac{\vec{\sigma}\cdot\mathbf{k}}{|\mathbf{k}|} = \vec{\sigma}\cdot\mathbf{k} \tag{131}$$

The starting point of the analysis of the reaction $\overline{p} + p \to e^+ + e^-$ is the standard expression of the matrix element in framework of one-photon exchange mechanism:

$$\mathcal{M} = \frac{e^2}{q^2}\overline{v}(k_2)\gamma_\mu u(k_1)\overline{u}(p_2)J_\mu v(p_1), \tag{132}$$

with

$$J_\mu = \left[F_1(q^2)\gamma_\mu - \frac{\sigma_{\mu\nu}q_\nu}{2M_p}F_2(q^2)\right] = \left[F_1(q^2)+F_2(q^2)\right]\gamma_\mu - \frac{(-p_1+p_2)_\mu}{2M_p}F_2(q^2),$$

where p_1, p_2, k_1 and k_2 are the four-momenta of initial antiproton and proton and the final electron and positron respectively, $q^2 > 4M_p^2$, $q = k_1 + k_2 = p_1 + p_2$. F_1 and F_2 are the Dirac and Pauli nucleon electromagnetic FFs, which are complex functions of the variable q^2 - in the TL region of momentum transfer.

In framework of one-photon exchange, the matrix element is written as the product of the leptonic and hadronic currents:

$$\mathcal{M} = \frac{e^2}{q^2}L_\mu J_\mu = \frac{e^2}{q^2}(L_0J_0 - \vec{L}\cdot\vec{J}) = -\frac{e^2}{q^2}\vec{L}\cdot\vec{J}, \tag{133}$$

where $L_0J_0 = 0$, due to the conservation of the leptonic and hadronic currents. The conservation of the current implies that $L\cdot q = 0$, i.e., $L_0q_0 - \vec{L}\cdot\vec{q} = 0$, but $\mathbf{q} = \mathbf{k_1} + \mathbf{k_2} = 0$ in CMS. Therefore, $L_0q_0 = 0$ for any energy q_0, i.e., $L_0 = 0$.

Let us reduce the expressions of the current in terms of σ (Pauli) matrices instead of Dirac γ matrices $J_\mu \to \varphi_2\tilde{J}_\mu\varphi_1$ (we keep in mind a global factor $(E+M_p)$).

$$\begin{aligned}
J_\mu &= (F_1+F_2)\left(\varphi_2,\ -\frac{\vec{\sigma}\cdot(-\mathbf{p})}{E+M_p}\varphi_2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\frac{\vec{\sigma}\cdot\mathbf{p}}{E+M_p}\varphi_1\\ \varphi_1\end{pmatrix}\\
&\quad+\left(\varphi_2,\ \frac{\vec{\sigma}\cdot(-\mathbf{p})}{E_1+M_p}\varphi_2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{2\mathbf{p}}{2M_p}F_2\begin{pmatrix}\frac{\vec{\sigma}\cdot\mathbf{p}}{E_1+M_p}\varphi_1\\ \varphi_1\end{pmatrix}\\
&= (F_1+F_2)\left(\varphi_2,\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E+M_p}\varphi_2\right)\begin{pmatrix}\vec{\sigma}\varphi_1\\ -\vec{\sigma}\frac{\vec{\sigma}\cdot\mathbf{p}}{E+M_p}\varphi_1\end{pmatrix}\\
&\quad+\frac{\mathbf{p}}{M_p}F_2\varphi_2\left(\frac{\vec{\sigma}\cdot\mathbf{p}}{E+m}+\frac{\vec{\sigma}\cdot\mathbf{p}}{E+M_p}\right)\varphi_1\\
&= (F_1+F_2)\left[\vec{\sigma} - \frac{1}{(E+M_p)^2}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\vec{\sigma}\cdot\mathbf{p}\right] + \frac{2\mathbf{p}}{M_p}F_2\varphi_2\frac{\vec{\sigma}\cdot\mathbf{p}}{E+M_p}\varphi_1.
\end{aligned} \tag{134}$$

Using the relation $p^2 = E^2 - M_p^2$, introducing the unit vectors $\hat{\mathbf{p}}$ and applying the following properties of σ matrices:

$$(2\hat{\mathbf{p}} - \vec{\sigma}\vec{\sigma}\cdot\hat{\mathbf{p}})\vec{\sigma}\cdot\hat{\mathbf{p}} = 2\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}} - \vec{\sigma},$$

one finds

$$\begin{aligned}
J_\mu &= (F_1+F_2)\left(\vec{\sigma} - 2\frac{E-M_p}{E+M_p}\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}} + \frac{E-M_p}{E+M_p}\vec{\sigma}\right) + \frac{2(E-M_p)}{M_p}F_2\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\\
&= (F_1+F_2)\left(\vec{\sigma} + \frac{E-M_p}{E+M_p}\vec{\sigma}\right) - 2\left[(F_1+F_2)\frac{E-M_p}{E+M_p} - \frac{E-M_p}{M_p}F_2\right]\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\\
&= \frac{2E}{E+M_p}(F_1+F_2)\vec{\sigma} - \frac{2(E-M_p)}{M_p(E+M_p)}[M_pF_1 + M_pF_2 - EF_2 - M_pF_2]\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\\
&= \frac{2E}{E+M_p}(F_1+F_2)\vec{\sigma} - 2E(F_1+F_2)\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}} + 2M_p\left(F_1 + \frac{E^2}{M_p^2}F_2\right)\\
&= \frac{2E}{E+M_p}\left[G_M(\vec{\sigma} - \hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}})\right] + 2M_pG_E\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}.
\end{aligned}$$

Finally (reminding the global factor) we find for the hadronic current:

$$\vec{J} = \sqrt{q^2}\varphi_2^\dagger\left[G_M(q^2)(\vec{\sigma} - \hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}) + \frac{1}{\sqrt{\tau}}G_E(q^2)\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}\right]\varphi_1, \tag{135}$$

where φ_1 and φ_2 are the two-component spinors of the antiproton and the proton, $\hat{\mathbf{p}}$ is the unit vector along the three momentum of the antiproton in CMS. The expression for the leptonic current is:

$$\vec{L} = \sqrt{q^2}\varphi_2^\dagger(\vec{\sigma} - \hat{\mathbf{k}}\vec{\sigma}\cdot\hat{\mathbf{k}})\varphi_1, \tag{136}$$

where $\varphi_1(\varphi_2)$ is the two-component spinor of the electron (positron), $\hat{\mathbf{k}}$ is the unit vector along the final electron three-momentum.

Note that Eq. (136) holds for the production of unpolarized lepton (sum over the lepton polarization). From this expression one can see the physical meaning of the particular relation between the nucleon electromagnetic FFs at threshold:

$$G_E(q^2) = G_M(q^2),\ q^2 = 4M_p^2.$$

The structure $\hat{\mathbf{p}}\vec{\sigma}\cdot\hat{\mathbf{p}}$ describes the $\overline{p}+p$ annihilation from D-wave, i.e., with angular momentum ℓ=2. At threshold, where $\tau\to 1$, the finite radius of the strong interaction allows only the S-state, and $G_M(q^2) - \frac{1}{\sqrt{\tau}}G_E(q^2) = 0$.

From Eqs. (133), (136), and (135) one can find the formulas for the unpolarized cross section, the angular asymmetry and all the polarization observables.

6.2. *The cross section*

To calculate the cross section when all particles are unpolarized, one has to sum over the polarization of the final particles and to average over the polarization of initial particles:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 q^2}\frac{|\mathbf{k}|}{|\mathbf{p}|},\ |\mathbf{k}| = \frac{\sqrt{q^2}}{2},\ |\mathbf{p}| = \sqrt{\frac{q^2}{4} - M_p^2},$$

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{e^4}{q^4}L_{ab}J_{ab},\ L_{ab} = L_aL_b^*,\ J_{ab} = J_aJ_b^*.$$
$$\overline{L_{ab}} = \overline{L_aL_b^*} \sim Tr(\sigma_a - \hat{k}_a\vec{\sigma}\cdot\mathbf{k})(\sigma_b - \hat{k}_b\vec{\sigma}\cdot\mathbf{k}) = 2(\delta_{ab} - k_ak_b). \tag{137}$$

Let us decompose the contribution to $\mathcal{M}$ in four terms classifying along FFs:

1) - $|G_M|^2$:

$$\frac{1}{2}Tr(\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})(\sigma_b - p_b\sigma\cdot p) =$$
$$\delta_{ab} - \sigma_ap_ap_b\vec{\sigma}\cdot\mathbf{p} - p_a\vec{\sigma}\cdot\mathbf{p}\sigma_b + p_ap_b\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\mathbf{p} = \delta_{ab} - p_ap_b. \tag{138}$$

Therefore $|G_M|^2$ contributes to the cross section with:

$$(\delta_{ab} - p_ap_b)(\delta_{ab} - k_ak_b) = \delta_{ab}\delta_{ab} - p^2 - k^2 - (\mathbf{p}\cdot\mathbf{k}) = 3 - 1 - 1 + \cos^2\theta. \tag{139}$$

2) - The term $G_EG_M^*$ vanishes:

$$\frac{1}{2}Tr(p_a\vec{\sigma}\cdot\mathbf{p}\sigma_b - p_ap_b\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\mathbf{p}) = \frac{1}{2}(p_ap_b - p_ap_b) = 0. \tag{140}$$

3) - The term $G_MG_E^*$ similarly vanishes:

$$\frac{1}{\tau}p_a\vec{\sigma}\cdot\mathbf{p}(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p}). \tag{141}$$

This shows that no interference term will be present in the cross section.

4) - $|G_E|^2$:

$$(\sigma_a - p_a\sigma\cdot p)(\sigma_b - p_b\sigma\cdot p) = \frac{1}{\sqrt{\tau}}\vec{\sigma}\cdot\mathbf{p}\frac{1}{\sqrt{\tau}}\vec{\sigma}\cdot\mathbf{p} = \frac{1}{\sqrt{\tau}}p_ap_b \tag{142}$$

Therefore $|G_E|^2$ contributes to the cross section with:

$$\frac{1}{\sqrt{\tau}}p_ap_b(\delta_{ab} - k_ak_b) = \frac{1}{\sqrt{\tau}}[1 - (\mathbf{p}\cdot\mathbf{k})^2] = \frac{1}{\tau}(1 - \cos^2\theta) = \frac{1}{\tau}\sin^2\theta. \tag{143}$$

We took into account the properties of σ matrices: $\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\mathbf{p} = p^2 = 1$, and $Tr\vec{\sigma}\cdot\vec{a}\vec{\sigma}\cdot\vec{b}\vec{\sigma}\cdot\vec{c} = i\vec{a}\cdot\vec{b}\times\vec{c}$.

Using the expressions (136) and (135), the formula for the cross section in CMS is:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \mathcal{N}\left[(1+\cos^2\theta)|G_M|^2 + \frac{1}{\tau}\sin^2\theta|G_E|^2\right], \tag{144}$$

where $\mathcal{N} = \dfrac{\alpha^2}{4\sqrt{q^2(q^2-4M_p^2)}}$, $\alpha = e^2/(4\pi) \simeq 1/137$, is a kinematical factor. This formula was firstly obtained in Ref.[22] Note that the normalization factor is inessential for the calculation of the polarization phenomena.

The angular dependence of the cross section, Eq. (144), results directly from the assumption of one-photon exchange, where the photon has spin 1 and the electromagnetic hadron interaction satisfies the $P-$invariance. Therefore, the measurement of the differential cross section at three angles (or more) would also allow to test the presence of 2γ exchange.

The electric and the magnetic FFs are weighted by different angular termss in the cross section, Eq. (144). One can define an angular asymmetry, $\mathcal{R}$, with respect to the differential cross section measured at $\theta = \pi/2$:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \sigma(\theta=\pi/2)\left[1+\mathcal{R}\cos^2\theta\right], \tag{145}$$

where $\mathcal{R}$ can be expressed as a function of FFs:

$$\mathcal{R} = \frac{\tau|G_M|^2 - |G_E|^2}{\tau|G_M|^2 + |G_E|^2}. \tag{146}$$

This observable should be very sensitive to the different underlying assumptions on FFs, therefore, a precise measurement of this quantity, which does not require polarized particles, would be very interesting. A deviation of the differential cross section from a linearity in $\cos^2\theta$ would be the signature of mechanisms beyond one photon exchange (similarly to a deviation form linearity in the Rosenbluth plot).

The q^2 dependence of the total cross section can be presented as follows:

$$\sigma(q^2) = \mathcal{N}\frac{8}{3}\pi\left[2|G_M|^2 + \frac{1}{\tau}|G_E|^2\right]. \tag{147}$$

6.3. *Polarization observables*

Polarization phenomena will be especially important in $\bar{p} + p \to \ell^+ + \ell^-$.

The dependence of the cross section on the polarizations $\vec{P}_1$ and $\vec{P}_2$ of the colliding antiproton and proton can be written as follows:

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)(\vec{P}_1,\vec{P}_2) = \left(\frac{d\sigma}{d\Omega}\right)_0 &[1 + A_y(P_{1y}+P_{2y}) + \\ &A_{xx}P_{1x}P_{2x} + A_{yy}P_{1y}P_{2y} + \\ &A_{zz}P_{1z}P_{2z} + A_{xz}(P_{1x}P_{2z}+P_{1z}P_{2x})],\end{aligned} \tag{148}$$

where the coefficients A_i and A_{ij} $(i,j = x,y,z)$, analyzing powers and correlation coefficients, depend on the nucleon FFs. Their explicit form is given below. The dependence (148) results from the P-invariance of hadron electrodynamics. The polarized hadronic tensor reads:

$$W_{ab}(\vec{P}_1,\vec{P}_2) = \frac{1}{2}TrJ_a\vec{\sigma}\cdot\vec{P}_1J_b^*\vec{\sigma}\cdot\vec{P}_2$$

and the cross section with unpolarized electrons is proportional to $L_{ab}\overline{W_{ab}}$.

6.4. *Single spin polarization observables*

In case of polarized antiproton beam with polarization $\vec{P}_1$, the contribution to the cross section can be calculated as:

$$\begin{aligned}&\left(\frac{d\sigma}{d\Omega}\right)_0\vec{A}_1 \sim -L_{ab}\frac{1}{4}TrJ_a\vec{\sigma}J_b^* = \\ &[(\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})G_M + \frac{1}{\tau}G_E p_a\vec{\sigma}\cdot\mathbf{p}](-\vec{\sigma}\cdot\vec{P}_1) \\ &[(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p})G_M^* + \frac{1}{\tau}G_E^* p_b\vec{\sigma}\cdot\mathbf{p}](\delta_{ab} - k_aK_b).\end{aligned} \tag{149}$$

(1) The term in $|G_M|^2$:

$$[1]: \ (\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p})\delta_{ab} - \tag{150}$$

$$[2]: \ (\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p})\hat{k}_a\hat{k}_b. \tag{151}$$

The first contribution (150) reduces to:

$$\begin{aligned}[1]: \quad & \sigma_a\vec{\sigma}\cdot\vec{P}_1\sigma_a - \sigma_a\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\sigma_a + p_a^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p} \\ &= -p_a(a\cdot P_1\times\mathbf{p} + p\cdot P_1\times\vec{a}) + p_a^2\vec{\sigma}\cdot\vec{P}_1 = 0.\end{aligned}$$

The second contribution (151) becomes:

$$\begin{aligned}[2]: \ & (\vec{\sigma}\cdot\mathbf{k} - \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1(\vec{\sigma}\cdot\mathbf{k} - \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}) \\ & \vec{\sigma}\cdot\mathbf{k}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k} - \vec{\sigma}\cdot\mathbf{k}\vec{\sigma}\cdot\vec{P}_1\mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p} - \\ & \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k} + (\mathbf{p}\cdot\mathbf{k})^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p} \\ & = -\cos\theta(\sigma\cdot\mathbf{k}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p} + \sigma\cdot\mathbf{p}\sigma\cdot\vec{P}_1\sigma\cdot\mathbf{k} \\ & = -\cos\theta[(\mathbf{k}\cdot\vec{P}_1\times\mathbf{p} + \mathbf{p}\cdot\vec{P}_1\times\mathbf{k}] = 0\end{aligned}$$

due to the antisymmetric terms in first parenthesis and the fact that the σ matrices have zero trace.

(2) The term $|G_E|^2$:

$$\frac{1}{\tau}\left[p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - (\mathbf{p}\cdot\mathbf{k})^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p}\right] = 0.$$

(3) The term $G_M G_E^*$

$$\begin{aligned}& \frac{1}{2}Tr\frac{1}{\tau}[(\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1 p_b\vec{\sigma}\cdot\mathbf{p}](\delta_{ab} - k_a k_b) \\ = & \frac{1}{\tau}[(\sigma_a - p_a\vec{\sigma}\cdot\mathbf{p})\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - (\vec{\sigma}\cdot\mathbf{k} - \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}].\end{aligned} \tag{152}$$

Let us decompose explicitly the components:

$$\begin{aligned}& \frac{1}{\tau}[(\sigma_x\vec{\sigma}\cdot\vec{P}_1 p_x\sigma_z + \sigma_y\vec{\sigma}\cdot\vec{P}_1 p_y\sigma_z) \\ & -(\sigma_x\sin\theta + \sigma_z\cos\theta - \sigma_z\cos\theta)\vec{\sigma}\cdot\vec{P}_1\cos\theta\sigma_z] \\ = & -\sigma_x\sin\theta\cos\theta\vec{\sigma}\cdot\vec{P}_1\sigma_z = -i\sin\theta\cos\theta P_{1y},\end{aligned}$$

$$\boxed{G_M G_E^* \to -i\sin\theta\cos\theta P_{1y}}$$

(4) Similarly for the term in $G_E G_M^*$ one finds:

$$\begin{aligned}& [p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1(\sigma_b - p_b\vec{\sigma}\cdot\mathbf{p})](\delta_{ab} - k_a k_b) \\ = & [p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}_a - p_a\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1 p_a\vec{\sigma}\cdot\mathbf{p} - \\ & \mathbf{p}\cdot\mathbf{k}\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{k} - (\mathbf{p}\cdot\mathbf{k})^2\vec{\sigma}\cdot\mathbf{p}\vec{\sigma}\cdot\vec{P}_1\vec{\sigma}\cdot\mathbf{p}] \\ = & i[p_a\vec{a}\cdot\mathbf{p}\times\vec{P}_1 - \cos\theta\mathbf{p}\cdot\vec{P}_1\times\mathbf{k}].\end{aligned}$$

Let us calculate the mixte product:

$$\vec{a}\cdot\mathbf{p}\times\vec{P}_1 \to p_x = p_y = 0; z p_z\times P_1 = 0.$$

More explicitly:

$$\left(\begin{array}{c|ccc|} p & 0 & 0 & 1 \\ P & P_{1x} & P_{1y} & P_{1z} \\ k & \sin\theta & 0 & \cos\theta \end{array}\right)$$

$$\boxed{G_E G_M^* \to \frac{i}{\sqrt{\tau}}\cos\theta\sin\theta P_{1y}}$$

In the calculation of the single spin polarization the terms related to $|G_E|^2$ and $|G_M|^2$ vanish. We add a global sign as the term for polarization of an antiparticle contains a "-" sign: $-\vec{\sigma}\cdot\mathbf{p}$.

For the interference terms, the only non zero analyzing power is related to the normal polarization P_y:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{1,y} = -\frac{i\mathcal{N}}{\sqrt{\tau}}\sin\theta\cos\theta[G_M G_E^* - G_E G_M^*] = \frac{\mathcal{N}}{\sqrt{\tau}}\sin 2\theta Im(G_M G_E^*). \tag{153}$$

Other observables can be obtained with some algebra in similar way. When the target is polarized, one writes:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 \vec{A}_2 = L_{ab}\frac{1}{4}TrJ_a J_b^*\vec{\sigma}.$$

Again the terms related to $|G_E|^2$ and $|G_M|^2$ vanish. Moreover, one can find $\vec{A}_2 = \vec{A}_1 = \vec{A}$.

Eq. (153) has been proved also in Ref.[22] One can see that this analyzing power, being T-odd, does not vanish in $\bar{p}+p \to \ell^+ + \ell^-$, even in one-photon approximation, due to the fact FFs are complex in time-like region. This is a principal difference with elastic ep scattering. Let us note also that the assumption $G_E = G_M$ implies $A_y = 0$, independently from any model taken for the calculation of FFs.

6.5. *Double spin polarization observables*

The contribution to the cross section, when both colliding particles are polarized is calculated through the following expression:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 A_{ab} = -\frac{1}{4}L_{mn}TrJ_m\sigma_a J_n^\dagger\sigma_b,$$

where a and $b = x, y, z$ refer to the $a(b)$ component of the projectile (target) polarization. Among the nine possible terms, $A_{xy} = A_{yx} = A_{zy} = A_{yz} = 0$, and the nonzero components are:

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_0 A_{xx} &= \sin^2\theta\left(|G_M|^2 + \frac{1}{\tau}|G_E|^2\right)\mathcal{N},\\
\left(\frac{d\sigma}{d\Omega}\right)_0 A_{yy} &= -\sin^2\theta\left(|G_M|^2 - \frac{1}{\tau}|G_E|^2\right)\mathcal{N},\\
\left(\frac{d\sigma}{d\Omega}\right)_0 A_{zz} &= \left[(1+\cos^2\theta)|G_M|^2 - \frac{1}{\tau}\sin^2\theta|G_E|^2\right]\mathcal{N},\\
\left(\frac{d\sigma}{d\Omega}\right)_0 A_{xz} &= \left(\frac{d\sigma}{d\Omega}\right)_0 A_{zx} = \frac{1}{\sqrt{\tau}}\sin 2\theta Re G_E G_M^*\mathcal{N}.
\end{aligned} \tag{154}$$

One can see that the double spin observables depend on the moduli squared of FFs, except A_{xz} (A_{zx}). Therefore, in order to determine the relative phase of FFs, in TL region, the interesting observables are A_y, and A_{xz}, which contain respectively the imaginary and the real part of the product $G_E G_M^*$.

7. Conclusion

We have given here a formal derivation of unpolarized cross section and polarization observables for the case of ep elastic scattering in the Breit system and $\bar{p}p$ annihilation into a (massless) lepton pair in CM system, where the calculation is simplified.

The results are model independent expressions of polarized and unpolarized experimental observables as functions of FFs, which hold in the assumption of one photon exchange mechanism taking into account the symmetries and the conservation laws of the electromagnetic and strong interactions.

Polarization observables play an important role as they are contain the interference of FFs, whereas only the moduli squared contribute to the unpolarized cross section.

The modelisation of the nucleon structure is contained in the parametrization of FFs. Different models have been developed in the recent years. In future, the interest will be focused on those models which can describe coherently all four nucleon FFs, proton and neutron, electric and magnetic, in SL and TL regions.

Precise data will strongly constrain nucleon models. Several experiments are planned or ongoing in electron accelerators as JLab, Mainz and colliders as Novosibirsk, BES, and Panda at FAIR. In SL region, the main purpose is to reach higher transferred momenta or better precisions. In TL region the individual determination of the electric and magnetic FFs at least in the region over threshold will be possible in next future. The measurements at the highest possible momentum transfer will allow to study asymptotic properties, where predictions exist from QCD and analyticity.

Search for effects beyond one photon exchange is object of a renewed experimental effort. The possibility to polarize antiprotons through spin filtering is also under investigation[61] opening the possibility to measure the relative phase of FFs in the time-like region.

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