

RADIATIVE CORRECTIONS TO LEPTON-HADRON INTERACTIONS

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A brief review of the present status of radiative corrections to processes of particle interaction is presented. Examples are given for QED corrections to processes of lepton-hadron and lepton-lepton interactions at intermediate and low energies. The method of electron structure function in the leading logarithmic approximation is described. The origin of Coulomb singularity in the final state interactions of charged particles is demonstrated. A relativized version of the Sommerfeld-Gamow-Sakharov factor is proposed. The vacuum polarization effect is discussed. General recommendations on application of radiative corrections in data analysis are given.

Keywords: .

1. Introduction

The aim of these lectures to review the effects due to QED *Radiative Corrections* (RC) being relevant for lepton-hadron interactions at (relatively) low energies. We will discuss typical contributions like vacuum polarization, Coulomb singularity in the final state interactions, the leading logarithmic approximation *etc.* Concrete examples on RC calculation will be given. The main attention will be paid to the features of the corrections themselves and on the application of them to the analysis of experimental data. So, we will not go into details of step-by-step calculations of the corrections. Instead we will concentrate on the discussion of particular features of RC which are relevant for construction of high-precision theoretical predictions and experimental data analysis.

By *radiative corrections* we mean quantum effects which contribute to observable quantities in higher orders in a coupling constant. In other words, RC are the effects which appear beyond the lowest order which will be called here the *Born approximation*. So the first step in treatment of any type of radiative corrections is the definition of the zeroth level. Typically the latter corresponds to a very limited number of tree-level Feynman diagrams describing the given process. But in certain cases we use the so-called *improved Born approximation* in which some simple more or less factorizable effects of higher orders are already included. In this case to avoid double counting one should exclude them from the RC contributions.

Sometimes radiative corrections can be confused with background processes which contribute to the same observable. Typically RC are just some modifications of the basic process, while background corresponds to other processes which only look similar to the basic one in the detector. But in a general situation one should solve a complex problem, where both RC and backgrounds are taken into account. Then the definition of each effect within the given problem should be done explicitly.

By corrections we usually mean small modifications. It is really so concerning RC in the bulk of realistic problems. But that is not *always* so. There are many examples of radiative corrections which are not small in comparison with the Born contribution. Sometimes we even have corrections of the order of several hundred percent. And below we will discuss a particular case in which the correction is infinitely large while the observable remains finite.

It is worth to mention also that radiative corrections is a subjects where interests of theoreticians and experimentalists are strongly connected. In order to improve the resulting precision in analysis of concrete experimental data, they should work in a tight collaboration. In many cases the numerical contribution of RC depends very much on the experimental conditions of particle registration.

The studies of radiative corrections are of ultimate importance for the modern elementary particles physics. In fact physics is a *natural science*, so it unifies experimental and theoretical investigations. On the other hand, physics belongs to *exact sciences* and now we deal with very accurate experiments and theoretical predictions. Let us assume that a pure experimental uncertainty

$$\delta^{exp} = \delta^{sys} \oplus \delta^{stat} \quad (1)$$

is the proper sum of systematical and statistical errors. Then the final result of the study, *i.e.* some physical quantity, would contain also a contribution due to theoretical uncertainties which always enter the game at a certain stage of the data analysis. In order not to spoil the accuracy of the experiment which has been obtained by considerable human and material investments, one should provide the theoretical error as small as possible. In practice it is highly desirable that the latter should satisfy the condition

$$\delta^{theor} \lesssim \delta^{exp}/3. \quad (2)$$

If the theoretical error would be of the same size as the experimental one then the resulting uncertainty will be $\delta^{theor \oplus exp} \approx 1.4 \cdot \delta^{exp}$ which will mean that that a huge part of the investments has been spent just for nothing. We see also that the precision of modern experiments is continuously growing up due to new hardware, improvement of analysis techniques, increasing of exposition time and so on.

All these facts show that the theoretical accuracy should be adequate to the experimental one. So we need more and more precise theoretical predictions which should include radiative corrections.

One could say that precise studies are required only for tests of the Standard Model (SM), while in searches for *New Physics* radiative corrections are not important at all. That is not true, of course. Really, we are always looking for new physical effects as a difference of an observable and the corresponding theoretical prediction. We are doing that at the edges of the explored field of physical phenomena. In this case theoretical predictions should be also as accurate as possible. This requirement is valid as for the new physics searches at very high energies of the Large Hadron Collider (LHC) as well as *e.g.* for studies of neutrinoless double beta decays at low energies.

There are two extreme possible scenarios of physical studies at the LHC: i) the most exiting case would be the discovery of many particles and new interactions; ii) nothing new is discovered besides the SM-like Higgs boson. Let us imagine the first situation. Very soon after the experimental discovery of new particles we will see in the literature a whole bunch of theoretical models pretending to describe the new phenomena. In this case only having very precise theoretical predictions received within the SM and in its extensions^a could help to choose the most adequate model of the new physics. In other words, the knowledge of radiative corrections will be required to discriminate different models of new physics. In the second case (which is obviously very unpleasant for the whole fundamental science) the physical program of experimental studies at the LHC will be shifted to the continuation of precision tests of the Standard Model and looking for possible (small) deviations from its predictions. In this case calculation of RC will be again of ultimate importance.

2. Types of Radiative Corrections

Let us discuss the typical types of radiative corrections. First of all, we subdivide RC into perturbative (computed order-by-order by means of expansion in a coupling constant) and non-perturbative which are found either phenomenologically from experimental data or from an exact solution of some equations in the quantum field theory, *e.g.* the Bethe-Salpeter ones. Sometimes we can also perform a resummation of a certain class of perturbative corrections, for example, that is usually done with the vacuum polarization effect in the propagator of a virtual photon, see Sect. 5. The main method in calculation of radiative corrections is expansion in a small parameter which can be: a coupling constant, transverse momentum, mass ratio *etc.*

RC are also usually separated according to the relevant type of interactions. So we can have QED, QCD and electroweak corrections. Note that separation of electroweak RC into pure QED and pure weak in the frames of the Standard Model is not always possible in a gauge invariant way. What is important that in practice we *always* have a mixture of RC of all types due to quantum effects. One of the primary problems in treatment of radiative corrections is to disentangle the mixture. Only in some extreme cases like the one of the electron anomalous magnetic moment we have almost a pure QED interaction.

There is a common statement that all relevant one-loop radiative corrections to any process being of interest for phenomenology have been computed long ago. That is not really so. Of course there is a lot of results in the literature. But for application for any new experiment we usually have to reconsider the evaluation of corrections in order to take into account concrete specific conditions.

As an example of radiative correction, let us look at the fit of the Higgs boson mass from LEP data, see Fig. 1 taken from Ref.¹

One can see that even so the the Higgs boson has not been observed at LEP (shadowed (yellow) region means $m_h > 114.4$ GeV), the data are sensitive to the Higgs mass value via radiative corrections (quantum loop effects). And if the Higgs boson is SM-like (no valuable contributions of new physics affected the data) than its mass is limited from above, $m_h < 144$ GeV, with a very large confidence level. With this respect one can remind the situation with the discovery of the top quark: well before

^aMany accurate predictions with one-loop RC have been already developed within minimal supersymmetric extensions of the SM.

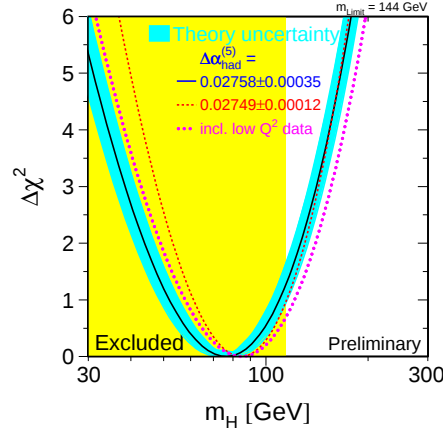


Fig. 1. Fit of the Higgs boson mass from LEP data.

the direct observation of this particle, its mass was fitted from the LEP data with high precision: $m_t(LEP) \approx 172 \text{ GeV}^b$. We hope that the LHC will soon find the Higgs boson and check whether its mass lies in the region allowed by the LEP data fit.

3. QED radiative corrections

Let us discuss the general features of QED RC. First of all those are the most typical corrections which are relevant for almost all observables in particle physics. It is well known that the main method of their calculation is the perturbative expansion in the fine structure constant $\alpha \approx \frac{1}{137}$. Looking at the details of this procedure one can conclude that the actual small parameter of the expansion is actually $\alpha/(2\pi)$. Numerical values of the first terms in this expansion

$$\frac{\alpha}{2\pi} \approx 0.12\%, \quad \left(\frac{\alpha}{2\pi}\right)^2 \approx 1.3 \cdot 10^{-4} \quad (3)$$

admit a fast convergence of the perturbative series. But in practice the situation is not so simple: in actual calculations there could be other important small and large parameters. For instance, at large energies we usually expand also in the series over the small ratio m/E , where m is the charged particle mass and E is its energy^c. An example of a large parameter in QED is the large logarithm $\ln(E^2/m^2)$, it will be discussed in detail in Sect. 6.

As an example of QED RC we can look at the small-angle Bhabha scattering process

$$e^-(k_1) + e^+(k_2) \rightarrow e^-(p_1) + e^+(p_2), \quad (4)$$

which was used for luminosity measurement at LEP. Table 1 shows values of various RC contributions δ_i in percent for different values of experimental cut x_c (see details of the set-up in Ref.²). The contribution of vacuum polarization (see Sect. 5) δ_{vp} depends on the momentum transferred in the process $\sqrt{Q^2} = \sqrt{-(p_1 - k_1)^2} \approx 1 \text{ GeV}$. One-photon contribution δ^γ (together with a part of δ_{vp}) correspond to $O(\alpha^1)$ term of the QED perturbative expansion. One can see that its numerical value is much more than the one expected from Eq. (3). The same observation can be done from the order of magnitude of the $O(\alpha^2)$ contributions $\delta_{LLA, NLO}^{2\gamma}$ and $\delta^{e^+e^-}$ due to photonic and pair corrections, respectively. The third order leading logarithmic photonic contribution $\delta^{3\gamma}$ is small compared to the experimental precision tag ~ 0.03 , but it is required to establish the theoretical uncertainty unambiguously. In spite of efforts of several groups of theoreticians, the latter was not reduced to the level adequate to the very accurate experimental measurement of small-angle Bhabha at LEP.³ Only recently complete two-loop results (neglecting small terms $\sim m_e^2/Q^2$) for RC to Bhabha scattering were obtained.⁴⁻⁶ Those hopefully will be used at the future e^+e^- International Linear Collider (ILC).

3.1. First order QED corrections

The first order QED radiative corrections is nowadays a rather standard ingredient in phenomenological applications of particle physics. Usually we decompose them into three parts: i) virtual (loop) corrections; ii) the ones due to a soft photon emission, and

^bDue to specific features of the SM, radiative corrections are much more sensitive to the top quark mass than to the Higgs boson one.

^cIn many (but not all) cases this ratio is squared.

x_c	δ_{VP}	δ^γ	$\delta_{LLA}^{2\gamma}$	$\delta_{NLO}^{2\gamma}$	$\delta^{e^+e^-}$	$\delta^{e^+e^-\gamma}$	$\delta^{3\gamma}$	$\sum \delta^i$
0.1	4.120	-8.918	0.657	0.162	-0.016	-0.017	-0.019	-4.031 ± 0.006
0.3	4.120	-9.626	0.615	0.148	-0.033	-0.008	-0.013	-4.797 ± 0.006
0.5	4.120	-10.850	0.539	0.129	-0.044	-0.003	-0.006	-6.115 ± 0.006
0.7	4.120	-13.770	0.379	0.130	-0.057	-0.001	0.005	-9.194 ± 0.006
0.9	4.120	-25.269	1.952	-0.085	-0.085	0.005	0.017	-19.379 ± 0.006

iii) the ones due to a single hard photon radiation:

$$\sigma^{1\text{-loop corrected}} = \sigma^{\text{Born}} + \sigma^{\text{Virt}}(\lambda) + \sigma^{\text{Soft}}(\lambda, \bar{\omega}) + \sigma^{\text{Hard}}(\bar{\omega}). \quad (5)$$

First of all this is done because analytically and numerically each part is calculated separately.

Decomposition (5) involves introduction of two auxiliary parameters: λ which regularizes the infrared singularity and $\bar{\omega}$ which separates the phase spaces of the hard and soft photons. The latter should be defined in a concrete reference frame, then the energy of a hard photon should be greater than $\bar{\omega}$:

$$\omega_{\text{hard}} > \bar{\omega}, \quad \omega_{\text{soft}} < \bar{\omega}, \quad \bar{\omega} \ll E. \quad (6)$$

Choosing the value of the soft-hard separator being small compared with the typical energy scale E (of the given process) we gain considerable simplifications in the calculations of the soft photon contribution to RC. For the infrared regulator λ we can take a fictitious photon mass with the conditions $\lambda \ll m$ and $\lambda \ll \bar{\omega}$.

Results for one loop QED corrections are known for very many processes. But sometimes in order to perform a concrete experimental study we have to re-compute the corrections taking into account specific experimental conditions. As an example we can take the classical process of bremsstrahlung off charged leptons, *e.g.* muons in collisions with heavy atoms:

$$\mu + A \rightarrow \mu + \gamma + A. \quad (7)$$

Some representatives of the relevant Feynman diagrams are given in Figs. 2 and 3. It appeared that existing calculations for RC to this process (known for about 50 years) are not suited for the COMPASS experiment conditions and we had to re-consider them.⁷

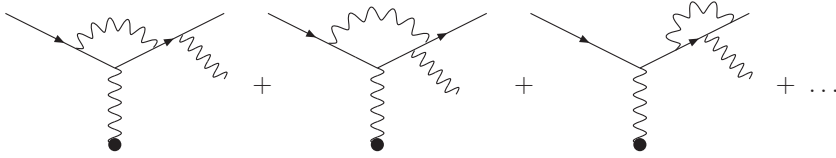


Fig. 2. Representatives of Feynman diagrams for virtual 1-loop RC to muon bremsstrahlung.

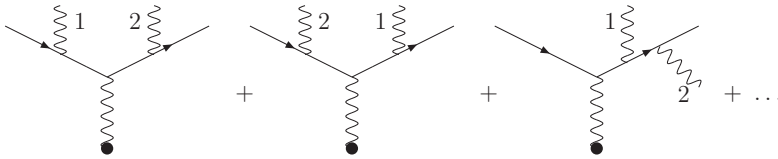


Fig. 3. Representatives of Feynman diagrams for real photon emission RC to muon bremsstrahlung.

The results of the calculations separated according to Eq. (5) are given in Table 2 and in Fig. 4.

ω/E_1	Born	Virtual	Soft ₁	Hard ₁	$\delta_1, \%$	Soft ₂	Hard ₂	$\delta_2, \%$
0.3	15677(1)	76.8(4)	- 260.1(1)	226.9(3)	+0.28	-307.0(1)	273.7(3)	+0.28
0.5	10836(1)	77.9(2)	- 319.0(1)	280.0(3)	+0.36	-377.4(1)	338.1(3)	+0.36
0.7	7337.7(1)	76.9(2)	- 363.3(1)	297.1(2)	+0.15	-430.9(1)	364.8(2)	+0.15
0.9	1267.4(1)	20.5(1)	- 111.1(2)	65.9(1)	-1.95	-132.4(2)	87.2(1)	-1.95

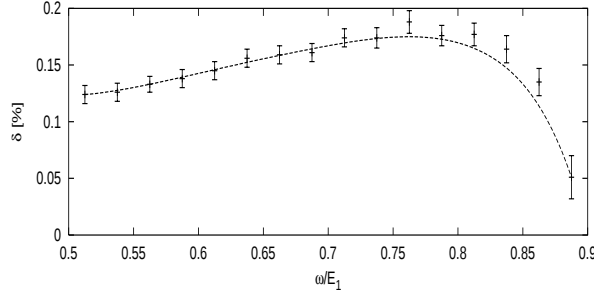


Fig. 4. Relative contribution of one-loop corrections for realistic set-up vs. the photon energy fraction.

In Fig. 4 one can see a peculiar feature: the size of the correction is rapidly increasing at large values of the observed photon energy fraction ω/E_1 . Emission of extra hard photons in this region is suppressed: we have an effective cut-off from above on the extra photon energy. In other words here we meet the typical situation: “the more we cut, the more we get” (when we cut-off a considerable part of hard radiation and get a large correction).

Another example of incompleteness of the RC studies is the calculation of the complete one-loop QED correction to the muon decay spectrum which was apparently finished only recently.^{8,9} The one-loop contribution to the spectrum reads (see⁸ for notation):

$$\frac{d^2\Gamma^{(1)}}{dxdc} = \Gamma_0 x^2 \beta \frac{\alpha}{2\pi} (f_1(x) + c\xi g_1(x)), \quad \beta = \sqrt{1 - \frac{m_e^2}{E_e^2}}, \quad (8)$$

$$\begin{aligned} f_1(x) = & f^{\text{Born}}(x) \left(\frac{2}{\beta} A + \frac{x^2(1-\beta^2) - 4(1+x\beta)}{2x\beta} \ln \frac{q^2}{m_\mu^2} \right. \\ & + \frac{4 - x^2(1-\beta^2)}{x\beta} \ln \frac{2 - x(1-\beta)}{2} \Big) + \frac{1}{\beta} \left(L + 2 \ln x + 2 \ln \frac{1+\beta}{2} \right) \left\{ \frac{5x^4}{384} (1-\beta^2)^3 \right. \\ & - \frac{x^3}{4} (1-\beta^2)^2 + \frac{3x^2}{32} (3 - 12\beta + \beta^2)(1-\beta^2) + x \left[\frac{2}{3} + 2\beta + (1-\beta^2) \left(\frac{3}{2} + \beta \right) \right] \\ & + \frac{1}{8} [-20 - 12\beta - 19(1-\beta^2)] + \frac{2}{x} + \frac{5}{6x^2} \Big\} + \left(\ln x + \ln \frac{1+\beta}{2} \right) \left[\frac{9}{4} x^2 (1-\beta^2) \right. \\ & + 2x(\beta^2 - 3) + 3 \Big] + f^{\text{Born}}(x) \left[-\frac{11}{18} x(1-\beta^2) + \frac{22}{27} \beta^2 - \frac{2}{9} \right] \\ & + x \left(-\frac{22}{27} \beta^4 + \frac{\beta^2}{2} - \frac{11}{6} \right) + \frac{22}{9} (3 - \beta^2) - \frac{22}{3x}, \\ A = & L \left(\ln \frac{q^2}{m_\mu^2} - \ln x + \ln \frac{1+\beta}{2\beta} + \ln \frac{2 - x(1-\beta)}{2\beta} \right) + \left[\ln \frac{q^2}{m_\mu^2} - 2 \ln x + 2 \ln \frac{1+\beta}{2} \right. \\ & + 4 \ln \frac{2 - x(1-\beta)}{2\beta} \Big] \left(\ln x + \ln \frac{1+\beta}{2} \right) + 2 \text{Li}_2 \left(\frac{(1-\beta)(2 - x(1+\beta))}{(1+\beta)(2 - x(1-\beta))} \right) \\ & - 2 \text{Li}_2 \left(\frac{2 - x(1+\beta)}{2 - x(1-\beta)} \right), \quad L \equiv \ln \frac{m_\mu^2}{m_e^2}, \\ f^{\text{Born}}(x) = & 3 - 2x + \frac{x}{4} (3x - 4)(1 - \beta^2), \quad \text{Li}_2(x) \equiv - \int_0^x \frac{dy}{y} \ln(1 - y). \end{aligned} \quad (9)$$

Here $x = 2E_e/m_\mu$ is the electron energy fraction and c is the cosine between the electron 3-momentum and the muon spin. Expression for $g_1(x)$ is similar, it can be found in Ref.⁸ In the above formula, one can find a large logarithm. Here the energy scale of the process is the muon mass, so the large log takes the form $L = \ln(m_\mu^2/m_e^2) \approx 11$. Numerically the terms enhanced by the large log give the bulk of the total one-loop correction.

Modern techniques of one-loop RC calculations involve automatized computer systems, *e.g.*: the Mathematica package for generation and visualization of Feynman diagrams **FeynArts**;¹⁰ the Mathematica package for algebraic calculations in elementary particle physics **FeynCalc**;¹¹ the package for evaluation of scalar and tensor one-loop integrals **LoopTools**;¹² the generic automated package for the calculation of Feynman diagrams at one-loop **GRACE-loop**;¹³ and other.

Among the computer systems used for one-loop correction calculations, there is the SANC project.¹⁴ The project is devoted to Support of Analytic and Numeric Calculations for experiments at colliders. It is accessible via the Internet [http://sanc.jinr.ru] and allows automatic computation of pseudo- and realistic observables with the one-loop precision for various processes of elementary particle interactions in the frames of the Standard Model. The theoretical basis of SANC is given by book,¹⁵ where one can find the detailed presentation of a consistent procedure of one-loop SM radiative correction calculations. The computer system computes SM predictions for a large number of processes of particle interactions and decays taking into account one-loop QED, QCD and electroweak RC.

3.2. Soft photon emission

Let us look at the process

$$p + \bar{p} \rightarrow e^+ + e^- + \gamma, \quad (10)$$

where the energy of the photon in the center-of-mass system is small $\omega < \bar{\omega} \ll E = \sqrt{s}/2$, where E is the total c.m.s. energy. Then the cross section of this process takes the factorized form: the Born level cross section of the non-radiative process $p + \bar{p} \rightarrow e^+ + e^-$ is multiplied by the so-called accompanying radiation factor:

$$\frac{d\sigma^{\text{Soft}}}{d\sigma^{\text{Born}}} = -\frac{4\pi\alpha}{(2\pi)^3} \int \frac{d^3k}{2\omega} \left(\frac{p_+}{p_+k} - \frac{p_-}{p_-k} + \frac{q_-}{q_-k} - \frac{q_+}{q_+k} \right)^2 = \frac{d\sigma_{\text{even}}^{\text{Soft}}}{d\sigma^{\text{Born}}} + \frac{d\sigma_{\text{odd}}^{\text{Soft}}}{d\sigma^{\text{Born}}}, \quad (11)$$

$$\begin{aligned} \frac{d\sigma_{\text{even}}^{\text{Soft}}}{d\sigma^{\text{Born}}} &= \frac{\alpha}{\pi} \left\{ -2 \left[\ln \left(\frac{2\bar{\omega}}{\lambda} \right) - \frac{1}{2\beta} L_\beta \right] - 2 \ln \left(\frac{\bar{\omega} \cdot m}{\lambda E} \right) \right. \\ &\quad \left. + 2 \frac{1+\beta^2}{2\beta} \left[\ln \left(\frac{2\bar{\omega}}{\lambda} \right) L_\beta - \frac{1}{4} L_\beta^2 + \Phi(\beta) \right] + 2 \left[\ln \left(\frac{2\bar{\omega}}{\lambda} \right) L_e - \frac{1}{4} L_e^2 - \frac{\pi^2}{6} \right] \right\}, \\ \Phi(\beta) &= \frac{\pi^2}{12} + L_\beta \ln \frac{1+\beta}{2\beta} + \ln \frac{2}{1+\beta} \ln(1-\beta) + \frac{1}{2} \ln^2(1+\beta) - \frac{1}{2} \ln^2 2 \\ &\quad - \text{Li}_2(\beta) + \text{Li}_2(-\beta) - \text{Li}_2 \left(\frac{1-\beta}{2} \right), \quad L_e \equiv \ln \frac{s}{m_e^2}, \\ \beta &= \frac{|\vec{p}_+|}{p_+^0} = \frac{|\vec{p}_-|}{p_-^0}, \quad L_\beta \equiv \ln \left(\frac{1+\beta}{1-\beta} \right). \end{aligned}$$

The P-odd contribution $d\sigma_{\text{odd}}^{\text{Soft}}$ and detailed notation can be found in Ref.¹⁶ In the formula above one can find the soft-hard separator $\bar{\omega}$ and the fictitious photon mass λ used to regularize the infrared singularity. Both parameters appear under the logarithm. We meet here again the large logarithm L_e which is about 16 at the threshold and grows logarithmically with the c.m.s. energy.

Factorization of the factor corresponding to the soft photon emission happens due to the difference of the energy scale of the two sub-processes: the hard non-radiative one and the one of the photon emission (from all charged particles in the initial and final states). An analytical calculation of a soft radiation factor is relatively simple. Moreover, going to higher order one can use the general result of the Yennie–Frautchi–Suura theorem.¹⁷ The latter claims that the factorization of multiple soft photon emission sub-processes happens not only with respect to the hard sub-process but also between different photon contributions. That allows to exponentiate the factor and thus find the effect re-summed in all orders of the perturbation theory.

Meanwhile, cancellation of the dependence on the auxiliary parameters λ and $\bar{\omega}$ should be provided. Cancellation of the terms containing $\ln \lambda$ happens in all orders in the sum of virtual (loop) and real soft photon emission contributions. That was proven in the well known Bloch–Nordsieck theorem. The dependence on the soft-hard separator should disappear after adding the hard photon contribution. The latter can be either calculated analytically or computed numerically. The numerical approach is nowadays the most appropriate because in this case it is possible to take into account all specific experimental conditions.

3.3. Hard photon emission

As an example of hard photon emission we can take the process

$$e^+(p_+) + e^-(p_-) \rightarrow \mu^+(q_+) + \mu^-(q_-) + \gamma(k). \quad (12)$$

Its differential cross section has the form¹⁸

$$d\sigma = \frac{\alpha^3}{2\pi^2 s^2} R d\Gamma, \quad d\Gamma = \frac{d^3k d^3q_- d^3q_+}{q_+^0 q_-^0 k^0} \delta(p_+ + p_- - q_- - q_+ - k),$$

$$R = \frac{s}{16} (4\pi\alpha)^{-3} \sum_{spins} |M|^2 = R_e + R_\mu + R_{e\mu},$$

where we separated the contributions due to the initial state radiation (ISR) R_e , the final state radiation (FSR) R_μ , and their interference $R_{e\mu}$. Let us first look at the ISR part:

$$R_e = \frac{s}{s_1^2} \left\{ \left[-\frac{m_e^2}{\chi_+^2} - \frac{m_e^2}{\chi_-^2} + \frac{s}{\chi_+ \chi_-} \right] \left(\frac{1}{2} tt_1 + \frac{1}{2} uu_1 + sm_\mu^2 \right) \right. \\ \left. + \frac{1}{\chi_-} (2m_\mu^2 \chi_+ - u_1 \chi'_+ - t_1 \chi'_-) + \frac{1}{\chi_+} (2m_\mu^2 \chi_- - u \chi'_- - t \chi'_+) \right. \\ \left. + P \left[q_-(u - t_1) + q_+(t - u_1) + 2m_\mu^2(p_+ - p_-) - \frac{p_-}{\chi_-} (2m_\mu^2 \chi_+ - u_1 \chi'_+ - t_1 \chi'_-) \right. \right. \\ \left. \left. + \frac{p_+}{\chi_+} (2m_\mu^2 \chi_- - u \chi'_- - t \chi'_+) \right] \right\}, \quad P = \frac{p_+}{\chi_+} - \frac{p_-}{\chi_-},$$

$$s = 2p_+ p_-, \quad s_1 = (q_+ + q_-)^2, \quad t = -2p_- q_-, \quad u = -2p_- q_+,$$

$$u_1 = -2p_+ q_-, \quad t_1 = -2p_+ q_+, \quad \chi_\pm = p_\pm k, \quad \chi'_\pm = q_\pm k.$$

The FSR and initial-final state radiation interference part are

$$R_\mu = A_\mu + B_\mu, \quad R_{e\mu} = A_{e\mu} + B_{e\mu},$$

$$A_\mu = -\frac{tt_1 + uu_1 + 2sm_\mu^2}{2s} \left(\frac{m_\mu^2}{(\chi'_+)^2} + \frac{m_\mu^2}{(\chi'_-)^2} - \frac{2q_- q_+}{\chi'_+ \chi'_-} \right),$$

$$A_{e\mu} = -\frac{tt_1 + uu_1 + 2sm_\mu^2}{2s_1} \left(\frac{t}{\chi_- \chi'_-} + \frac{t_1}{\chi_+ \chi'_+} - \frac{u}{\chi_- \chi'_+} - \frac{u_1}{\chi_+ \chi'_-} \right),$$

$$B_\mu = \frac{1}{s} \left\{ -\frac{4m_\mu^2 \chi_+ \chi_-}{\chi'_+ \chi'_-} - \frac{t_1 \chi_- + u \chi_+}{\chi'_-} - \frac{u_1 \chi_- + t \chi_+}{\chi'_+} \right. \\ \left. + Q[p_+(t - u) + p_-(u_1 - t_1) + \frac{q_-}{\chi'_-} (t_1 \chi_- + u \chi_+) - \frac{q_+}{\chi'_+} (u_1 \chi_- + t \chi_+)] \right\},$$

$$B_{e\mu} = -\frac{2}{s_1} \left\{ -t - t_1 + u + u_1 - \frac{1}{2} u_1 \left(\frac{p_-}{\chi_-} - \frac{q_+}{\chi'_+} \right) (Q \chi'_+ + P \chi_-) \right. \\ \left. - \frac{1}{2} t_1 \left(\frac{p_-}{\chi_-} - \frac{q_-}{\chi'_-} \right) (Q \chi'_- - P \chi_-) - m_\mu^2 (\chi_+ + \chi_-) (QP) \right. \\ \left. + \left(\frac{m_\mu^2}{\chi'_+} - \frac{m_\mu^2}{\chi'_-} \right) (\chi_- - \chi_+) - \frac{1}{2} t \left(\frac{q_+}{\chi'_+} - \frac{p_+}{\chi_+} \right) (Q \chi'_+ - P \chi_+) \right. \\ \left. - \frac{1}{2} u \left(\frac{q_-}{\chi'_-} - \frac{p_+}{\chi_+} \right) (Q \chi'_- + P \chi_+) \right\},$$

$$P = \frac{p_+}{\chi_+} - \frac{p_-}{\chi_-}, \quad Q = \frac{q_-}{\chi'_-} - \frac{q_+}{\chi'_+}.$$

Note that the so-called *collinear singularities* appear in the denominators $\chi_\pm$ and $\chi'_\pm$. They are regularized by the fermion masses. In fact, $\chi_- = p_- k = p_-^0 \omega [1 - \beta_- \cos(\vec{p}_- \cdot \vec{k})]$, where $\beta_- = \sqrt{1 - m_e^2/(p_-^0)^2} < 1$ and so on.

4. The Coulomb Singularity

It is well known that the electromagnetic interaction between charged particles in the final state can considerably affect the observable reaction rate. For example, the cross section of electron-positron annihilation into a proton-anti-proton pair becomes different from zero at the threshold due to the final state interactions.¹⁹ Another observable effect is the difference in energy behavior at the threshold of the annihilation channels with production of charged and neutral mesons, see e.g. Ref.²⁰

If the relative velocity of the charged particles is small ($v \ll 1$), then the effect of multiple photon exchange between them becomes significant. This fact has been discussed in the literature for a long time. It was shown already in the textbook

by A. Sommerfeld²¹ that the correction due to re-scattering of charged particles in the final state is proportional to the bound state wave function at the origin squared, $|\Psi(0)|^2$, see also book.²² So that the scattering (or annihilation) channel acquires some features of the corresponding bound state. G. Gamow has shown²³ that the same factor is relevant for the description of the Coulomb barrier in nuclear interactions. Using the non-relativistic Schrödinger equation, A. Sakharov derived this factor for the case of charged pair production²⁴ in the form

$$T(v) = \frac{\eta(v)}{1 - e^{-\eta(v)}}, \quad \eta(v) = \frac{2\pi\alpha}{v}, \quad (13)$$

where v is the non-relativistic relative velocity of the particles in the created pair, and $\alpha \approx 1/137$ is the fine structure constant. This function will be called below as the Sommerfeld-Gamow-Sakharov (SGS) factor.

For practical applications of the factor for modern experiments, it is highly desirable to have a relativized version of the SGS factor. This problem and some other ways of generalization of the factor, *e.g.* for non-equal masses and P -waves, is under discussion in the literature for a long time, see papers^{25–28,30,31} and references therein.

Let us look at the explicit formula⁶ for the FSR correction to the processes $e^+ + e^- \rightarrow \pi^+ + \pi^-$:

$$\begin{aligned} \sigma_{\pi\pi(\gamma)}^0 &= \frac{\pi\alpha^2}{3s} \beta_\pi^3 |F_\pi(s)|^2 |1 - \Pi(s)|^2 \left(1 + \frac{\alpha}{\pi} \Lambda(s)\right), \\ \Lambda(s) &= \frac{1 + \beta_\pi^2}{\beta_\pi} \left\{ 4\text{Li}_2\left(\frac{1 - \beta_\pi}{1 + \beta_\pi}\right) + 2\text{Li}_2\left(-\frac{1 - \beta_\pi}{1 + \beta_\pi}\right) - \left[3\ln\left(\frac{2}{1 + \beta_\pi}\right) + 2\ln\beta_\pi \right] \ln\frac{1 + \beta_\pi}{1 - \beta_\pi} \right\} \\ &\quad - 3\ln\left(\frac{4}{1 - \beta_\pi^2}\right) - 4\ln\beta_\pi + \frac{1}{\beta_\pi^3} \left[\frac{5}{4}(1 + \beta_\pi^2)^2 - 2 \right] \ln\frac{1 + \beta_\pi}{1 - \beta_\pi} + \frac{3}{2} \frac{1 + \beta_\pi^2}{\beta_\pi^2}, \end{aligned}$$

where β_π is the pion velocity in the c.m.s. frame. Some Feynman diagrams for $\mathcal{O}(\alpha)$ corrections to this process are shown in Fig. 5. One can see that the radiative correction $\Lambda(s)$ is proportional to the inverse power of β_π . It is important that this Coulomb singularity appears only in the correction, but the total corrected cross section is not singular for $\beta_\pi \rightarrow 0$.

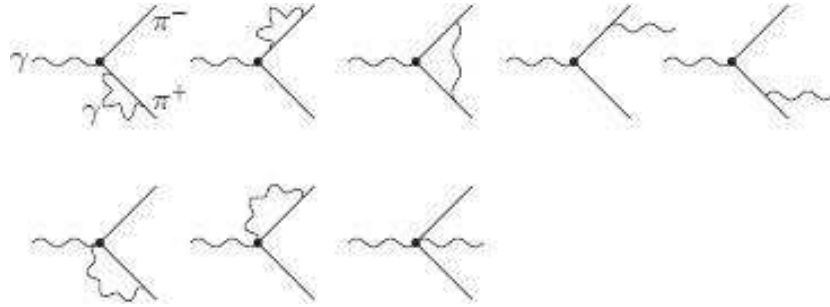


Fig. 5. Representatives of Feynman diagrams contribution to one-loop QED FSR corrections in $e^+e^- \pi\pi^+\pi^-$.

Let us consider the case of the final state interaction of charged particles produced close to the threshold, *e.g.* in electron-positron annihilation

$$e^-(k_1) + e^+(k_2) \rightarrow a^-(p_1) + a^+(p_2), \quad (14)$$

$$s = (k_1 + k_2)^2 = (p_1 + p_2)^2 \gtrsim (m_1 + m_2)^2, \quad (15)$$

where $a^\pm$ can be scalar, spinor, or vector particles. The Born-level cross section σ^{Born} of this process depends on the type of interaction and the spin. But in any case in the center-of-mass system, it is proportional to the first power of factor $\beta_{1,2}$ which comes from the phase space volume and vanishes at the threshold $s \rightarrow (m_1 + m_2)^2$,

$$\begin{aligned} \beta_{1,2} &= \frac{2p}{p_1^0 + p_2^0}, \quad p \equiv |\vec{p}_1| = |\vec{p}_2| = \frac{\sqrt{\Lambda(s, m_1^2, m_2^2)}}{2\sqrt{s}}, \\ p_1^0 + p_2^0 &= 2\sqrt{s}, \quad \Lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \end{aligned} \quad (16)$$

For the case of equal masses this factor takes the usual form of the relativistic velocity $\beta = \sqrt{1 - m^2/(p^0)^2}$ of the final state particles.

The relativistic relative velocity of our particles is

$$v_{\text{rel}} = \frac{\sqrt{\Lambda(s, m_1^2, m_2^2)}}{s - m_1^2 - m_2^2} = \frac{\sqrt{[s - (m_1^2 + m_2^2)][s - (m_1^2 - m_2^2)]}}{s - m_1^2 - m_2^2}, \quad (17)$$

which is a relativistic invariant, $0 < v_{\text{rel}} < 1$ (we work in the natural units $\hbar = c = 1$).

In the one-loop approximation the QED radiative correction to the annihilation cross section gets contributions from virtual and real photon emission:

$$\sigma^{1\text{-loop}} = \sigma^{\text{Born}} \left(1 + \delta^{\text{Virt}} + \delta^{\text{Real}} \right). \quad (18)$$

The last term in the parentheses is proportional to β^2 , it is strongly suppressed at the threshold. Explicit expressions for different contributions with the exact dependence on the final state particle mass (for the equal mass case) can be found *e.g.* in Ref.³² for fermions and in Ref.³³ for scalars. The final state one-loop virtual correction (in the on-mass-shell renormalization scheme) is described by the triangle diagram shown in Fig. 6. There are three types of integrals over the loop momentum: the scalar, the vector and the tensor ones:

$$\{I_S, I_V^\mu, I_T^{\mu\nu}\} = \int \frac{d^4k}{i\pi^2} \frac{\{1, k^\mu, k^{\mu\nu}\}}{((p_1 + k)^2 - m_1^2 + i\varepsilon)((p_2 - k)^2 - m_2^2 + i\varepsilon)(k^2 + i\varepsilon)}.$$

The tensor one contains an ultraviolet divergence, which has to be removed by the standard renormalization procedure. The vector integral I_V^μ is finite. The scalar integral I_S contains an infrared divergence, which cancels out in the sum with the contribution of the real final state radiation.

Fig. 6. Feynman diagram for one-loop virtual correction in the final state.

Direct calculations show that the contributions of the vector and tensor integrals are proportional at least to the second power of final state particle velocities. While the scalar integral reveals at the threshold the well known Coulomb singularity, it is proportional to $1/\sqrt{\Lambda(s, m_1^2, m_2^2)}$. The coefficients before the integrals depend in general on the type of the particles, but the one standing at the scalar one is *universal*, it is the same for scalar, spinor and vector final state particles. The contributions of the one-loop scalar integral to the cross section can be presented in the form

$$\delta\sigma_S^{1\text{-loop}} = \sigma^{\text{Born}} \frac{\alpha}{2\pi} (s - m_1^2 - m_2^2) C_0(m_1^2, m_2^2, s, m_1^2, m_2^2), \quad (19)$$

where the triangular scalar one-loop integral C_0 is written in the LoopTools package¹² notation. So, by comparison of the first order of the perturbation theory in the limit $s \rightarrow (m_1 + m_2)^2$ with the corresponding term in the expansion of the SGS factor, we can adjust the parameter of the latter. It appears that the proper choice is just the substitution of the non-relativistic relative velocity by the relativistic one, see also Ref.³⁴ This choice is also supported by studies performed within relativistic quasipotential models.^{28,35}

There is one important point here. In the description of the final state interactions we meet the situation when we have both perturbative and non-perturbative contributions. In such a case it is easy to get the so-called *double counting*. Really, the expansion in α of the SGS factor gives a part of the terms which are also included in the perturbative results. To avoid the double counting we introduce a scheme of *matching* between the two results. Here it can be done in the following way:

$$\sigma^{\text{Corr.}} = \sigma^{\text{Born}} \left(T(v) - \frac{\pi\alpha}{v} - \frac{\pi^2\alpha^2}{3v^2} - \dots \right) + \Delta\sigma^{1\text{-loop}} + \Delta\sigma^{2\text{-loop}} + \dots \quad (20)$$

where $\Delta\sigma^{n\text{-loop}}$ is the n -th order perturbative contribution to the observed corrected cross section $\sigma^{\text{Corr.}}$.

5. Vacuum polarization

A photon in QED can create a pair of charged particle and anti-particle. This pair can be either virtual (it annihilates soon after the creation) or real if the photon is off mass-shell and energy-momentum conservation law permits the "photon decay". The effect arising due to virtual pair creation is called in QED as *vacuum polarization* (VP), see Fig. 7.

Perturbative QED describes this effect for the case of lepton pairs. Results up to the fourth order in α are known. Let us look at the first order contributions. In the space like region (when the photon 4-momentum squared is negative: $q^2 < 0$)

$$\Pi_\ell(q^2) = -\frac{\alpha}{9\pi} \left(5 - 12\eta + 3(-1 + 2\eta) \sqrt{1 + 4\eta} \ln \frac{\sqrt{1 + 4\eta} + 1}{\sqrt{1 + 4\eta} - 1} \right) + O(\alpha^2),$$

where $\eta \equiv m_\ell^2/(-q^2)$ and m_ℓ is the lepton mass. In the time-like region below the threshold of real pair production ($4m_\ell^2 > q^2 > 0$) we have

$$\Pi_\ell(q^2) = -\frac{\alpha}{9\pi} \left(5 - 12\eta + 3(-1 + 2\eta) \sqrt{-1 - 4\eta} \arctan \frac{\sqrt{-1 - 4\eta}}{-1 - 2\eta} \right) + O(\alpha^2).$$

Above the threshold ($q^2 > 4m_\ell^2$) we get a nonzero imaginary part:

$$\begin{aligned} \Pi_\ell(q^2) = & -\frac{\alpha}{9\pi} \left(5 - 12\eta + 3(-1 + 2\eta) \sqrt{1 + 4\eta} \ln \frac{1 + \sqrt{1 + 4\eta}}{1 - \sqrt{1 + 4\eta}} \right) \\ & -i \frac{\alpha}{3} (1 - 2\eta) \sqrt{1 + 4\eta} + O(\alpha^2). \end{aligned}$$

So the leptonic contribution to VP is under control, the uncertainty in its calculation is negligible.

For high energies ($q^2 \gg m_\ell^2$) we get

$$\Pi(q^2) \approx -\frac{\alpha}{3\pi} \sum_f Q_f^2 N_c^f \left(\ln \frac{q^2}{m_\ell^2} - \frac{5}{3} \right), \quad (21)$$

where we sum over all fermions (f) with electric charge Q_f and number of colors N_c^f . The above formula describes both lepton and quark contributions. For the latter we require also $q^2 \gg \Lambda_{QCD}$.

Resummation of the geometric progression

$$1 + \Pi(q^2) + \Pi^2(q^2) + \dots = \frac{1}{1 - \Pi(q^2)} \quad (22)$$

gives us the conventional expression for the running QED coupling constant

$$\alpha(q^2) = \frac{\alpha}{1 - \text{Re}\Pi(q^2)}. \quad (23)$$

Here $\alpha(0) \equiv \alpha = 137.035999084(51)$ is the value of the fine structure constant extracted from experimental data at $q^2 \rightarrow 0$. The condition $\Pi(0) = 0$ is just the choice of the renormalization point within the so called *on-mass-shell scheme*. It is not unique, remind that at high energies we often use schemes where the point is shifted, for example in the $\alpha(M_Z^2)$ scheme we fix the value of the coupling constant at the Z boson mass. The latter scheme was convenient for applications at LEP experiments. Another popular choice is the G_{Fermi} (or G_μ) scheme in which the coupling constant is (re)normalized from very accurate measurements of the muon lifetime.³⁶

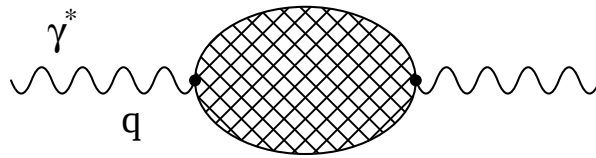


Fig. 7. Vacuum polarization in photon propagator.

Hadronic contribution to vacuum polarization Π_{had} is hard to compute starting the QCD Lagrangian (lattice simulations are far from the required precision). However, it is possible to use the optical theorem to obtain the real part of $\Pi_{\text{had}}(q^2)$ from the imaginary part. In fact in the time like region

$$\text{Im}\Pi_{\text{had}}(q^2) \sim \sigma(e^+e^- \rightarrow \text{hadrons}) \quad (24)$$

The dispersion integral is then given by

$$\Delta\alpha_{\text{had}}^{(5)}(q^2) = -\frac{q^2}{4\pi^2\alpha} \text{P} \int_{m_\pi^2}^{\infty} \frac{\sigma_{\text{had}}^0(s) ds}{s - q^2}$$

Five quark flavors are taken into account (we include e^+e^- annihilation channels into the corresponding mesons and baryons)^d. The lower limit of the integral corresponds to the lowest energy channel of hadron production: $e^+ + e^- \rightarrow \pi^0\gamma$. This process has a rather small cross section, but we should take it into account to reach the required high precision in the description of vacuum polarization. In general, VP contributes to practically all observables in particle physics. Knowing it is very important for construction of theoretical predictions for the anomalous magnetic moment of muon, to extract the Higgs boson mass, see Fig. 1, *etc.*

For practical applications we use a combination of analytical results for leptonic contributions with phenomenological parameterizations of the hadronic effects. Several different computer codes are available (for a discussion and comparison see Ref.³⁷): REPI³⁸

[<http://hbu.web.cern.ch/hbu/aqed/aqed.html>]

(only for the space-like region); function HADR5N by F. Egerlehner³⁹

[<http://www-com.physik.hu-berlin.de/~fjeger/>];

parametrization by the CMD collaboration (Novosibirsk)

[<http://cmd.inp.nsk.su/~ignatov/vpl/>];

and the HNMT routine.⁴⁰ Fig. 8 shows the dependence of the quantity $|1 + \Pi(s)|^2$ as a function of $\sqrt{s} = \sqrt{|q^2|}$, where positive $\sqrt{s}$ correspond to the time-like region ($q^2 > 0$) and negative $\sqrt{s}$ formally define the space like case ($q^2 < 0$). In the time like region one can clearly see resonance peaks of vector mesons: ρ , ω , ϕ , J/Ψ and so on. The solid line shows the sum of leptonic and hadronic contributions. The dotted one represents the pure leptonic effect.

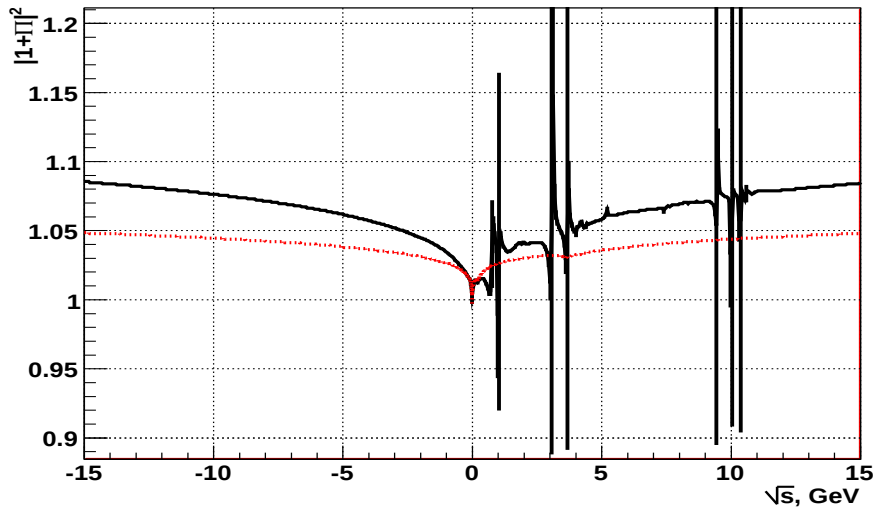


Fig. 8. $|1 + \Pi|^2$ from CMD-2 compilation for space- and time-like momenta (labelled $\sqrt{s}$); solid (black) lines: leptonic plus hadronic contributions, dotted (red) lines: only leptonic contributions.

6. The Leading Logarithmic Approximation

Calculations of radiative corrections in QED for processes with characteristic energies being large compared with the electron mass, $E \gg m_e$, reveal the following peculiar property. It appears that besides the expansion in α it becomes very useful to expand also in the small parameter m_e^2/E^2 and in the large parameter $L \equiv \ln(E^2/m_e^2)$. Usually the terms suppressed by the small parameter can be neglected in RC (it is sufficient to keep them only in the Born level cross section). And the terms enhanced by the so called *large logarithm* L give the bulk of the correction value. Calculation of these terms can be performed by specific methods which

^dThe top quark and $W^\pm$ boson contributions are computed separately.

are much simpler than the general ones (see Sect. 3) developed for complete calculations of perturbative corrections. Moreover, higher order leading logarithmic (*leading log*) corrections can be re-summed.

Let us discuss the expansion in $\mathcal{O}(\alpha^n L^n)$, $n = 0, 1, 2, \dots$, where

$$L \equiv \ln(\Lambda^2/m_e^2) \quad (25)$$

and Λ is the so-called *factorization scale*, $\Lambda^2 \gg m_e^2$. Typical values of Λ will be the center-of-mass energy for annihilation processes and the momentum transfer for scattering one. But in general this quantity is an parameter which can be varied.

The construction of the leading log approximation is based on the fundamental feature of quantum mechanics: interference of amplitudes which happen at different energy scales (or distances) is suppressed. In other words, we know that amplitudes of small-distance and large-distance sub-processes factorize with respect to each other. And neglecting their interference we get the final result for an observable quantity as a product of corresponding probabilities.

Let us look at the process $a + b \rightarrow c + d + \gamma$ with emission of a hard photon at a small angle with respect to the momentum of the initial particle a . Direct calculations show that its cross section can be presented, see Ref.¹⁸ for details, in the form

$$\begin{aligned} d\sigma[a(p_1) + b(p_2) \rightarrow c(q_1) + d(q_2) + \gamma(k \approx (1-z)p_1)] \\ \approx d\hat{\sigma}[a(zp_1) + b(p_2) \rightarrow c(q_1) + d(q_2)] \otimes R_H^{\text{ISR}}(z) \\ R_H^{\text{ISR}}(z) = \frac{\alpha}{2\pi} \left[\frac{1+z^2}{1-z} \left(\ln \frac{E^2}{m^2} - 1 + l_0 \right) + 1 - z + \mathcal{O}\left(\frac{m^2}{E^2}\right) + \mathcal{O}(\vartheta_0^2) \right], \\ \vartheta_\gamma = \widehat{\vec{p}_1 \vec{k}} < \vartheta_0, \quad m/E \ll \vartheta_0 \ll 1, \quad l_0 = \ln(\vartheta_0^2/4), \end{aligned} \quad (26)$$

where we integrated over the emission angles limited from above by the small parameter ϑ_0 . Here $\otimes$ denotes the so-called *convolution* operation, which is actually the integration over the photon energy fraction $z \equiv k^0/p_1^0$. The large logarithm appears in the *collinear radiation factor* R_H^{ISR} as the logarithm of the initial particle energy $E = p_1^0$ over its mass m . Factorization in the above formula appeared as the result of direct calculations of Feynman diagrams where small terms proportional to ϑ^2 and m^2/E^2 were neglected.

The amplitudes describing soft photon emission obviously do not interfere with the ones for hard radiation because their phase space do not overlap. So the one-loop correction to a differential cross section can be presented as

$$d\sigma = d\sigma^{\text{Soft+Virt}}(\Delta) + d\sigma^{\text{Hard(coll)}}(\Delta, \vartheta_0) + d\sigma^{\text{Hard(non-coll)}}(\Delta, \vartheta_0), \quad (27)$$

where [Soft] and hard collinear [Hard(coll)] photon contributions are factorized at the Born cross section. The contribution $d\sigma^{\text{Hard(non-coll)}}$ takes into account radiation of photons at large angles ($\vartheta > \vartheta_0$) and does not contain any large log. The dependence on the auxiliary parameters $\Delta \ll 1$ and ϑ_0 cancels out in the sum of contributions.

In this way, direct calculations in many cases explicitly demonstrate factorization properties of terms proportional to the large logs. What is important for us is that such observations can be generalized for a wide class of processes and for all orders of the perturbation theory. The *factorization theorems* were proved in QCD,^{41,42} they can be easily adapted for QED.

The energy scale in the large log is an effective separator of the scales in the hard (short-distance) and soft (long-distance) sub-processes. In the example of collinear photon emission considered above changing of the large log scale corresponds to variation of the auxiliary parameter ϑ_0 , while the sum (27) remains unchanged. In the general case the total result should not depend on the *factorization scale*. This property is due to the conformal symmetry of the given theory. Of course particle masses (and Λ_{QCD}) break the symmetry, but for large energies the conformal properties are restored. For this reason we can apply very powerful methods of the renormalization group approach, see review⁴³ and references therein.

So one can write renormalization group equations for application in QCD and QED at large energies. In particular the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP)^{44–48} evolution equation for the non-singlet electron structure function has the form

$$\mathcal{D}^{\text{NS}}(z, Q^2) = \delta(1-z) + \int_{m^2}^{Q^2} \frac{\alpha(q^2)}{2\pi} \frac{dq^2}{q^2} \int_z^1 \frac{dx}{x} P^{(0)}(x) \mathcal{D}^{\text{NS}}\left(\frac{z}{x}, q^2\right),$$

where m is the electron mass; $P^{(0)}$ is the first order non-singlet *splitting function*; $\alpha(q^2)$ is the QED running coupling constant. Here we are going to consider only the electron contribution to vacuum polarization:

$$\alpha(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{q^2}{m^2}}. \quad (28)$$

Structure function $\mathcal{D}^{\text{NS}}(z, Q^2)$ is the probability density to find an electron with energy fraction z in the given electron if the energy scale is limited by $\sqrt{Q^2}$ from above.

The splitting function

$$P^{(0)}(z) \equiv P_{ee}(z) = \left[\frac{1+z^2}{1-z} \right]_+ = \frac{1+z^2}{1-z} - \delta(1-z) \int_0^1 dx \frac{1+x^2}{1-x} \quad (29)$$

is used to describe the transition from an electron e into another electron e with energy fraction z with respect to the initial one. The singularity in this function at $z \rightarrow 1$ is regularized by means of the so-called *plus prescription* which is defined by

$$\int_{x_{\min}}^1 dx [f(x)]_+ g(x) = \int_0^1 dx f(x) [g(x) \Theta(x - x_{\min}) - g(1)]. \quad (30)$$

The complete set of evolution equations in the leading logarithmic approximation (LLA) of QED reads

$$\begin{aligned} \mathcal{D}_{ee}(x, s) &= \delta(1-x) + \int_{m^2}^s \frac{d\alpha(t)}{2\pi t} \left[\int_x^1 \frac{dy}{y} \mathcal{D}_{ee}(y, t) P_{ee}\left(\frac{x}{y}\right) + \int_x^1 \frac{dy}{y} \mathcal{D}_{\gamma e}(y, t) P_{e\gamma}\left(\frac{x}{y}\right) \right], \\ \mathcal{D}_{\bar{e}\bar{e}}(x, s) &= \int_{m^2}^s \frac{d\alpha(t)}{2\pi t} \left[\int_x^1 \frac{dy}{y} \mathcal{D}_{\bar{e}\bar{e}}(y, t) P_{\bar{e}\bar{e}}\left(\frac{x}{y}\right) + \int_x^1 \frac{dy}{y} \mathcal{D}_{\gamma \bar{e}}(y, t) P_{\bar{e}\gamma}\left(\frac{x}{y}\right) \right], \\ \mathcal{D}_{\gamma e}(x, s) &= -\frac{2}{3} \int_{m^2}^s \frac{d\alpha(t)}{2\pi t} \mathcal{D}_{\gamma e}(x, s) \\ &\quad + \int_{m^2}^s \frac{d\alpha(t)}{2\pi t} \left[\int_x^1 \frac{dy}{y} \mathcal{D}_{ee}(y, t) P_{\gamma e}\left(\frac{x}{y}\right) + \int_x^1 \frac{dy}{y} \mathcal{D}_{\bar{e}\bar{e}}(y, t) P_{\gamma \bar{e}}\left(\frac{x}{y}\right) \right]. \end{aligned}$$

Thanks to the smallness of α_{QED} , solutions of the evolution equations in QED can be obtained by iterations. The initial conditions for LLA QED are simple, for instance, one can take just the delta function as the zeroth approximation for $\mathcal{D}_{ee}(x, s)$. Note that in QCD the structure functions can not be computed analytically starting from the first principles and having only the parameters of the SM as input. Instead, we extract QCD structure function from experimental data. But the evolution of the functions with respect to the factorization scale is described by means of the DGLAP equations as in QED as well as in QCD.

Iteration and further application of the structure functions involves the *convolution* operation. Let us take two functions $f(x)$ and $g(y)$ defined for $0 \leq x, y \leq 1$. Their convolution is given by

$$[f \otimes g](z) = \int_0^1 dx \int_0^1 dy \delta(z - xy) f(x) g(y) = \int_z^1 \frac{dx}{x} f(x) g\left(\frac{z}{x}\right) \quad (31)$$

where $0 \leq z \leq 1$. This operation can be generalized for special functions regularized by plus-prescription:

$$\begin{aligned} \left[[f]_+ \otimes [g]_+ \right](z) &= \lim_{\Delta \rightarrow 0} \left\{ \int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x} f_{\Theta}(x) g_{\Theta}\left(\frac{z}{x}\right) + f_{\Delta} g_{\Theta}(z) + f_{\Theta}(z) g_{\Delta} \right\}, \\ f_{\Delta} &= - \int_0^{1-\Delta} dx f(x), \quad f_{\Theta}(x) = f(x) \Big|_{x < 1}. \end{aligned} \quad (32)$$

It is accepted to distinguish so-called *singlet* (S) and *non-singlet* (NS) parts of the electron structure function $\mathcal{D}_{ee}$:

$$\mathcal{D}_{ee} = \mathcal{D}^S + \mathcal{D}^{\text{NS}}, \quad \mathcal{D}^S = \mathcal{D}_{\bar{e}\bar{e}}. \quad (33)$$

In the Feynman diagram language the non-singlet part corresponds to the graphs with continuous electron line, while in the singlet case the registered electron with energy fraction z is a component of a created e^+e^- pair (it is connected with the initial electron via a photon line).

Non-singlet structure functions have the properties

$$\begin{aligned} \int_0^1 \mathcal{D}^{NS}(x, \beta) dx &= 1, \quad \beta \equiv \frac{2\alpha}{\pi} L, \\ \int_x^1 \frac{dy}{y} \mathcal{D}^{NS, \gamma}(y, \beta_1) \mathcal{D}^{NS, \gamma}\left(\frac{x}{y}, \beta_2\right) &= \mathcal{D}^{NS, \gamma}(x, \beta_1 + \beta_2). \end{aligned} \quad (34)$$

Let us now discuss the so-called *master formula* which shows how to apply the electron structure function method for evaluation of QED RC to a given process. Actually we take this formula from QCD. For concreteness let us take Bhabha scattering, where we have electrons and positrons both in the final and initial states. We will use the QCD-like massless partons. In QED there are three type of partons: electrons, positrons and photons. According to the factorization theorems, the corrected cross section of this process at high energies ($E \gg m_e$) can be presented in the form

$$\begin{aligned} d\sigma &= \sum_{a,b,c,d} \int_{z_1}^1 dz_1 \int_{z_2}^1 dz_2 \mathcal{D}_{ae}^{\text{str}}(z_1) \mathcal{D}_{be}^{\text{str}}(z_2) \left(d\sigma_{ab \rightarrow cd}^{\text{Born}}(z_1, z_2) + d\bar{\sigma}^{(1)}(z_1, z_2) + O(\alpha^2 L^0) \right) \\ &\quad \times \int_{\bar{y}_1}^1 \frac{dy_1}{Y_1} \int_{\bar{y}_2}^1 \frac{dy_2}{Y_2} \mathcal{D}_{ec}^{\text{frg}}\left(\frac{y_1}{Y_1}\right) \mathcal{D}_{ed}^{\text{frg}}\left(\frac{y_2}{Y_2}\right), \\ \mathcal{D}_{ee}^{\text{str, frg}}(z) &= \delta(1-z) + \frac{\alpha}{2\pi} d_1(z, \mu_0, m_e) + \frac{\alpha}{2\pi} LP^{(0)}(z) \\ &\quad + \left(\frac{\alpha}{2\pi}\right)^2 \left(\frac{1}{2} L^2 P^{(0)} \otimes P^{(0)}(z) + LP^{(0)} \otimes d_1(z, \mu_0, m_e) + LP_{ee}^{(1; \gamma, \text{pair}) \text{str, frg}}(z) \right) \\ &\quad + O(\alpha^2 L^0, \alpha^3), \quad L \equiv \ln \frac{Q^2}{\mu_0^2}, \quad d\bar{\sigma}^{(1)} = d\sigma^{(1)} \Big|_{m_e=0, \overline{\text{MS}}} \\ P^{(0)}(z) &= \left[\frac{1+z^2}{1-z} \right]_+, \quad d_1(z, \mu_0, m_e) = \left[\frac{1+z^2}{1-z} \left(\ln \frac{\mu_0^2}{m_e^2} - 2 \ln(1-z) - 1 \right) \right]_+. \end{aligned}$$

Here $P_{ee}^{(1; \gamma, \text{pair}) \text{str, frg}}(z)$ are the next-to-leading order (NLO) splitting functions, we also *borrow* them from QCD (by reduction to the abelian case), see Refs.^{49,50} Function d_1 defines the initial condition for the evolution of electron structure and fragmentation functions, μ_0 is the renormalization scale, in QED we choose it usually to be equal to the electron mass: $\mu_0 = m_e$.

The structure of the master formula is as follows. The electron structure functions $\mathcal{D}_{ae}^{\text{str}}(z_1)$ and $\mathcal{D}_{be}^{\text{str}}(z_2)$ give the probability density for transition of the initial massive electron and positron into massless partons of type a and b , respectively. The differential cross sections $d\sigma_{ab \rightarrow cd}^{\text{Born}}$ and $d\bar{\sigma}^{(1)}$ describe the parton-level process $a + b \rightarrow c + d$ at the Born and one-loop level, respectively. They are usually called *coefficient functions* which are dependent on the process, while all other elements of the master formula are universal (they are the same for a wide class of QED processes). The fragmentation functions $\mathcal{D}_{ec}^{\text{frg}}$ and $\mathcal{D}_{ed}^{\text{frg}}$ give the probability density for conversion of the massless partons c and d into massive electron and positron, respectively. The one-loop cross section $d\bar{\sigma}^{(1)}$ computed for massless particles is divergent. It should be regularized. Here we apply the modified minimal subtraction scheme⁵¹ ($\overline{\text{MS}}$). In this way the The sum over all possible intermediate parton reactions is taken. The master formula can be expanded in α . The first two terms in this expansion reproduce the complete Born and one-loop expressions for the Bhabha scattering process (terms proportional to m_e^2/E_e^2 are neglected). Higher order corrections produced by this formula define the leading and next-to-leading log contributions.

It is worth to note that the above formula suppose integration over the angular phase space of secondary (emitted) particles. But experimental conditions do not allow perform such integration if certain cuts are imposed. In this case a more detailed study of kinematics should be performed,⁵² see below Sect. 6.3.

6.1. Logarithmic approximation: examples

Let us return to the example of Bhabha scattering and look at the numerical effect in logarithmic corrections. Here we will look in the expansion the powers of the large log of the sum of virtual and soft photonic corrections in $O(\alpha^2)$, where the analytic result is known.⁴ So we can compare in Fig. 9 the LLA = $O(\alpha^2 L^2)$ (solid lines), NLO = $O(\alpha^2 L^1)$ and NNLO = $O(\alpha^2 L^0)$ contributions for $\Delta = 1$ and $\sqrt{s} = 100$ GeV, see Ref.⁵³ for details. Relative contributions of the terms proportional to the large log to i -th power

$$r_i^{(2)}(\theta) = \frac{d\sigma^{\text{Soft+Virt}}[L^i]/d\theta}{d\sigma^{\text{Born}}/d\theta} \quad (35)$$

depend on the choice of the factorizations scale Λ . On the left plot we have $\Lambda = \sqrt{s}$, *i.e.* the scale is equal to the center-of-mass energy of the process. In this case the contributions have comparable magnitudes, even so that they are ordered according to the power of the large log. On the right plot we choose $\Lambda = \sqrt{-t}$, *i.e.* the scale is equal to the momentum transferred. It appears that in this case the hierarchy of the contributions is very strong: the LLA contributions dominates, the next-to-leading one is small, and the NNLO one is almost invisible. So we see that the proper choice of the factorization scale is very important. The choice $\Lambda = \sqrt{-t}$ for Bhabha scattering is justified by the fact that the t -channel exchange dominates in this process.

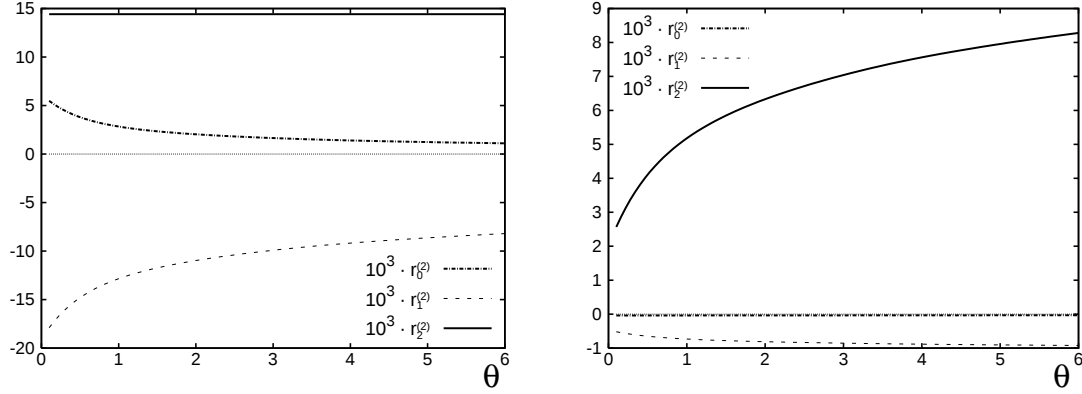


Fig. 9. The dependence on the choice of the factorization scale: $\Lambda = \sqrt{s}$ (left) and $\Lambda = \sqrt{-t}$ (right).

Let us look now at radiative corrections to the muon decay spectrum. In this process we have just one fast charged particle in the final state, so the application of the electron structure function method is simplified. The differential decay width of the anti-muon decay can be written as

$$\frac{d^2\Gamma^{\mu^+ \rightarrow e^+ \nu \bar{\nu}}}{dz d\cos\theta} = \Gamma_0 \left[F(z) - \cos\theta P_\mu G(z) \right] + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right),$$

$$\Gamma_0 \equiv \frac{G_\mu^2 m_\mu^5}{192\pi^3} \left(1 + \frac{3}{5} \frac{m_\mu^2}{m_W^2} \right), \quad z \equiv \frac{2E}{m_\mu},$$

Function $F(z)$ and $G(z)$ can be expanded in α and $L \equiv \ln(m_\mu^2/m_e^2)$ as

$$\begin{aligned} F(z) &= f_{\text{Bom}}(z) + \frac{\alpha}{2\pi} f_1(z) + \left(\frac{\alpha}{2\pi}\right)^2 f_2(z) + \mathcal{O}(\alpha^3), \\ f_1(z) &= L^1 \cdot f_1^{\text{LL}}(z) + L^0 \cdot f_1^{\text{NLO}}(z), \\ f_2(z) &= L^2 \cdot f_2^{\text{LL}}(z) + L^1 \cdot f_2^{\text{NLO}}(z) + L^0 \cdot f_2^{\text{NNLO}}(z) \end{aligned} \quad (36)$$

and so on. The Born level contributions are simple:

$$f_{\text{Bom}}(z) \equiv f_0(z) = z^2(3 - 2z), \quad g_{\text{Bom}}(z) \equiv g_0(z) = z^2(1 - 2z).$$

The first order corrections enhanced by large logs can be obtained by convolution of the Born level functions with the lowest order splitting function:

$$\begin{aligned} f_1^{\text{LL}}(z) &= P^{(0)}(\bullet) \otimes f_0(z) = \frac{5}{6} + 2z - 4z^2 + \frac{8}{3}z^3 + 2z^2(3 - 2z) \ln \frac{1-z}{z} P^{(0)}(\bullet) \otimes f_0(z), \\ g_1^{\text{LL}}(z) &= P^{(0)}(\bullet) \otimes g_0(z) = -\frac{1}{6} - 4z^2 + \frac{8}{3}z^3 + 2z^2(1 - 2z) \ln \frac{1-z}{z}. \end{aligned} \quad (37)$$

One can see that they agree with the corresponding terms received in the direct calculations,^{8,54} see also Eq. (9).

Using the second order terms in the electron structure function we get the $O(\alpha^2 L^2)$ photonic RC to the spectrum:⁵⁵

$$\begin{aligned}
 f_2^{\text{LL}(\gamma)}(z) &= P^{(0)}(\bullet) \otimes P^{(0)}(\bullet) \otimes f_0(z) \\
 &= 4z^2(3 - 2z)\Phi(z) + \left(\frac{10}{3} + 8z - 16z^2 + \frac{32}{3}z^3\right)\ln(1 - z) \\
 &\quad + \left(-\frac{5}{6} - 2z + 8z^2 - \frac{32}{3}z^3\right)\ln z + \frac{11}{36} + \frac{17}{6}z + \frac{8}{3}z^2 - \frac{32}{9}z^3, \\
 g_2^{\text{LL}(\gamma)}(z) &= 4z^2(1 - 2z)\Phi(z) + \left(-\frac{2}{3} - 16z^2 + \frac{32}{3}z^3\right)\ln(1 - z) \\
 &\quad + \left(\frac{1}{6} + 8z^2 - \frac{32}{3}z^3\right)\ln z - \frac{7}{36} - \frac{7}{6}z + \frac{8}{3}z^2 - \frac{32}{9}z^3, \\
 \Phi(z) &\equiv \text{Li}_2\left(\frac{z-1}{z}\right) + \ln^2 \frac{1-z}{z} - \frac{\pi^2}{6}.
 \end{aligned} \tag{38}$$

By the second order photonic RC we mean here the contributions of Feynman diagrams with two photons. Each of them can be either real or virtual. Besides the photonic RC in the second order there are also pair corrections, their leading log contributions can be computed in the same way, see.^{50,55}

6.2. Kinoshita-Lee-Nauenberg theorem

The large logs discussed above are divergent for $m_e \rightarrow 0$, in other words, they are a kind of *mass singularities*. Due to intrinsic (hidden) conformal properties of QED and QCD these mass singularities have a tendency to cancel out in inclusive observable quantities. The Kinoshita-Lee-Nauenberg (KLN) theorem^{56,57} defines the conditions when the coefficients before the large logs do vanish. In practice we usually see that the large log terms from virtual loop corrections cancel out the corresponding terms coming from real radiation.

Let us formulate the theorem in the following way. If we *can not* or just *do not* distinguish the final states of a pure electron and of a combination of the electron with accompanying it photon(s), *i.e.* the energy and momentum of the *registered* electron is the sum of the energies and momenta of the bare electron and the photons $E_e^{\text{observed}} = E_e + \Sigma E_\gamma$, then the large logarithm corresponding to collinear photon emission (see Eq. (26)) do not appear in the final answer. In the electron structure function approach the cancellation of the large logs is provided by the first property of the non-singlet function in Eq. (34).

In practice, cancellation of large logs happens for FSR corrections to (sufficiently) *inclusive observables* and for *calorimetric* electron registration. The large logs which come from the initial state radiation usually do not cancel out, since the KLN theorem conditions for them are not fulfilled. Also the large logs which appear in the correction due to vacuum polarization (in the QED running coupling constant) do not cancel out.

6.3. Matching of LLA with $O(\alpha)$ RC

Having both complete $O(\alpha)$ and LLA in $O(\alpha^n L^n)$ ($n = 1, 2, \dots$) one should avoid the *double counting*. In fact, the first order correction already contains all terms of the order $O(\alpha^1 L^1)$. The procedure which allows keep the correct first order result while adding the leading log corrections will be called here as *matching*. A possibility of such a matching is represented by the master formula (35). But as discussed above, the latter is valid only for sufficiently inclusive observables. Here we will show an example how to disentangle the double counting keeping at the same moment the possibility to impose experimental cuts on radiative events.

Let us consider the process of electron-positron annihilation into muons. First order corrections due to real hard photon emission in this process were discussed in Sect. 3.3. Adding of the higher order leading log terms provided by the electron

structure function approach to the known Born and complete first order corrections can be done by the following formula:

$$\begin{aligned}
 \frac{d\sigma^{e^+e^- \rightarrow \mu^+\mu^-(\gamma)}}{d\Omega_-} &= \int_{z_{\min}}^1 \int_{z_{\min}}^1 dz_1 dz_2 \frac{\mathcal{D}(z_1, s) \mathcal{D}(z_2, s)}{|1 - \Pi(sz_1 z_2)|^2} \frac{d\tilde{\sigma}_0(z_1, z_2)}{d\Omega_-} \left(1 + \frac{\alpha}{\pi} K\right) \\
 &+ \left\{ \frac{\alpha^3}{2\pi^2 s^2} \int_{\substack{k^0 > \bar{\omega} \\ \hat{k} p_{\pm} > \theta_0}} \frac{R_e|_{m_e=0}}{|1 - \Pi(s_1)|^2} \frac{d\Gamma}{d\Omega_-} + \frac{D}{|1 - \Pi(s_1)|^2} \right\} \\
 &+ \left\{ \frac{\alpha^3}{2\pi^2 s^2} \int_{k^0 > \bar{\omega}} \left(\text{Re} \frac{R_{e\mu}}{(1 - \Pi(s_1))(1 - \Pi(s))^*} + \frac{R_{\mu}}{|1 - \Pi(s)|^2} \right) \frac{d\Gamma}{d\Omega_-} \right. \\
 &\left. + \text{Re} \frac{C_{e\mu}}{(1 - \Pi(s_1))(1 - \Pi(s))^*} + \frac{C_{\mu}}{|1 - \Pi(s)|^2} \right\}, \quad (39) \\
 C_{\mu} &= \frac{2\alpha}{\pi} \frac{d\sigma_0}{d\Omega_-} \ln \frac{\bar{\omega}}{E} \left(\frac{1 + \beta^2}{2\beta} \ln \frac{1 + \beta}{1 - \beta} - 1 \right), \quad z_{\min} = \frac{2m_{\mu}}{2E - m_{\mu}}, \\
 C_{e\mu} &= \frac{4\alpha}{\pi} \frac{d\sigma_0}{d\Omega_-} \ln \frac{\bar{\omega}}{E} \ln \frac{1 - \beta c}{1 + \beta c}, \quad K = K_{\text{odd}} + K_{\text{even}},
 \end{aligned}$$

see notation and other details in Ref.¹⁸ The condition $\hat{k} p_{\pm} > \theta_0$ exclude the kinematical domain of collinear photon emission from the initial electron and positron. So, the corresponding large logs do not appear from the integral of the matrix element. They are coming, instead, from the structure functions. K is the so-called K -factor, here it comes from virtual and soft photon corrections. The large logs which appear in this part of the correction are also removed. But it was explicitly demonstrated that the proper amount of large logs is restored by the structure functions also here.

7. General Remarks on RC

Let us summarize the present status of radiative correction calculations and discuss the general ways of their application.

- 1 Many analytical results are in the literature for QED, QCD and electroweak RC within the Standard Model. Radiative corrections have been studied in models beyond SM, like in the Minimal Supersymmetric SM or in the Chiral Perturbation Theory (an effective models for low energy strong interactions).
- 2 Advanced techniques of multi-loop and multi-leg diagrams calculation have been developed.
- 3 But still application of (even) well known results to a concrete case is rather non-trivial:
 - old analytic calculations can have obsolete approximations,
 - different effects should be combined properly,
 - experimental conditions should be taken into account.
- 4 Semi-analytic codes like *e.g.* ZFITTER⁵⁸ and HECTOR⁵⁹ are well suited for inclusion of different effects.
- 5 But the best way is to incorporate RC into Monte Carlo event generators. This task is not simple because RC typically have kinematics (and dynamics) being much more complicated than the one of the Born approximation.
- 6 Dedicated Monte Carlo codes developed to describe a specific process are potentially more suitable for consistent inclusion of radiative corrections and interplay of other sub-leading effects that the General purpose MC programs like PYTHIA, HERWIG, PHOTOS *etc.*

So in these lectures, we have discussed general properties of radiative corrections. Most of the examples were given for the pure QED RC, but the QCD and electroweak corrections have very similar features. As we have seen above, radiative corrections could be as very small as well as very large. Only after a careful study of a particular process taking into account the conditions of the corresponding experiment one may get an idea about the magnitude of RC in the give case. We discussed also certain methods which help to extract the numerically most important contributions enhanced by large logarithms. Knowing the general features of radiative corrections should be also useful in application of existing ready-to-use solutions, *e.g.* computer codes or analytic formulae, to concrete problems in particle physics.

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