The Dirac Equation

Elementary Particle Physics

Strong Interaction Fenomenology

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Relativistic equation to describe the electron (including its spin).
Conservation of probability $\Rightarrow$ linearity in the time derivatives
Relativistic invariance $\Rightarrow$ linearity in the space derivatives

$$i\frac{\partial\psi}{\partial t}=\left[-i\left(\alpha_1\frac{\partial}{\partial x^1}+\alpha_2\frac{\partial}{\partial x^2}+\alpha_3\frac{\partial}{\partial x^3}\right)+\beta m\right]\psi$$

$$-\frac{\partial^2\psi}{\partial t^2}=\left(-\vec{\nabla}^2+m^2\right)\psi$$

$$i\frac{\partial}{\partial t}=-i\left(\alpha_1\frac{\partial}{\partial x^1}+\alpha_2\frac{\partial}{\partial x^2}+\alpha_3\frac{\partial}{\partial x^3}\right)+\beta m$$

$$\alpha_i^2=\beta^2=1$$

$$\alpha_i\alpha_j+\alpha_j\alpha_i=0 \quad i\neq j$$

$$\alpha_i\beta+\beta\alpha_i=0$$

Massless Fermions

Term with β is absent

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}$$

$$\boxed{\alpha_i = -\sigma_i}$$

σ_i Pauli matrices

$$\boxed{i \frac{\partial \psi}{\partial t} = \vec{\sigma} \cdot \vec{p}\, \psi}$$

ψ two-component spinor

Fermions with non-zero Mass

The smallest matrices for which there is a solution are 4×4 matrices.

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha_1 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \qquad \beta = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}$$

The γ Matrices

Since we have 4x4 matrices with 16 elements, there are 16 independent ones. We can write the results in terms of any set that satisfies the conditions in slide 2. A convenient set is given by the so-called γ matrices.

$$\gamma^i = \beta\alpha_i \quad \gamma^0 = \beta$$

$$\gamma^\mu = \left(\gamma^0; \gamma^i\right)$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$i\frac{\partial\psi}{\partial t} = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

$$\boxed{\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0}$$

$\psi^\dagger \gamma_\mu \psi$ is of the form $\begin{pmatrix} . & . & . & . \end{pmatrix} \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \begin{pmatrix} . \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} . & . & . & . \end{pmatrix} \begin{pmatrix} . \\ . \\ . \\ . \end{pmatrix} = \text{number}$

$$\overline{\psi} = \psi^\dagger \gamma^0$$

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$\overline{\psi}\gamma^\mu \psi$ scalar in spin space, transforms as a four-vector
$\overline{\psi}\psi$ trasforms as a scalar
$\overline{\psi}\gamma^5 \psi$ trasforms as a pseudoscalar (odd under parity)
$\overline{\psi}\gamma^5 \gamma^\mu \psi$ trasforms as an axial four-vector (even under parity)

Properties of the γ Matrices

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$$

$$\gamma^{0\dagger} = \gamma^{0}$$

$$\gamma^{k\dagger} = \left(\beta\alpha^{k}\right)^{\dagger} = \alpha^{k\dagger}\beta$$

$$= \alpha^{k}\beta = -\beta\alpha^{k} = -\gamma^{k}$$

$$\left(\gamma^{0}\right)^{2} = \gamma^{0}\gamma^{0} = 1$$

$$\gamma^{k}\gamma^{k} = \beta\alpha^{k}\beta\alpha^{k} = -1$$

Currents

$$i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^k \frac{\partial \psi}{\partial x^k} - m\psi = 0$$

Dirac Equation

$$-i\frac{\partial \psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial \psi^\dagger}{\partial x^k}\gamma^k - m\psi^\dagger = 0$$

Hermitian Conjugate

$$\partial_\mu \left(\overline{\psi}\gamma^\mu \psi\right) = 0$$

$$\boxed{j^\mu = \overline{\psi}\gamma^\mu \psi}$$

Conserved Current

For example, for electrically charged fermions:

$$\boxed{j_\mu^{electric} = -e\overline{\psi}\gamma_\mu \psi}$$

Consider as an example a free particle of four-momentum p^μ, described by a plane wave. The Dirac equation becomes:

$$\left(\gamma^\mu p_\mu - m\right)\psi = 0$$

$$\overline{\psi}\left(\gamma^\mu p_\mu - m\right) = 0$$ Hermitian Conjugate

We obtain:

$$\boxed{\overline{\psi}\gamma^\mu\psi = \frac{p^\mu}{m}\overline{\psi}\psi}$$

Interpreting $\overline{\psi}\psi$ as a probability density, $\overline{\psi}\gamma^\mu\psi$ is what one would expect for a free-particle current.

Free Particle Solutions

$$\gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We assume $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ where ψ_R e ψ_L are two-component spinors

$$\left(\gamma^\mu p_\mu - m\right)\psi = 0 \qquad \begin{pmatrix} -m & p_0 + \vec{\sigma}\cdot\vec{p} \\ p_0 - \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = 0$$

$$\boxed{\begin{array}{l} -m\psi_R + \left(p_0 + \vec{\sigma}\cdot\vec{p}\right)\psi_L = 0 \\ \left(p_0 - \vec{\sigma}\cdot\vec{p}\right)\psi_R - m\psi_L = 0 \end{array}}$$

(a) These equations can be written as:

$$\psi_R = \left(\frac{p_0 + \vec{\sigma} \cdot \vec{p}}{m} \right) \psi_L$$

$$\psi_L = \left(\frac{p_0 - \vec{\sigma} \cdot \vec{p}}{m} \right) \psi_R$$

(b) Solutions exist for positive or negative p_0 and can be interchanged by $\psi_L \leftrightarrow -\psi_R$.

(c) If $m=0$ the two equations separate. Since $\vec{\sigma} \cdot \vec{p}$ clearly measures the component of spin along the direction of motion, ψ_R is the large solution for $\vec{\sigma} \cdot \vec{p} > 0$ and p_0 positivo or $\vec{\sigma} \cdot \vec{p} < 0$ and p_0 negative, with ψ_L having the opposite correspondence.

$\vec{\sigma} \cdot \vec{p}$ helicity. For massless or relativistic particles $\vec{\sigma} \cdot \vec{p} \big/ p^0 \cong \vec{\sigma} \cdot \hat{p}$

L represents a left-handed positive energy solution, *R* a right-handed one.
For a massless or relativistic particle, for a left-handed state with $p_0>0$ $\psi_L \ll \psi_R$.

(d) If $m \neq 0$ the two equations do not separate. In particular we will see that a mass term in a Lagrangian can be interpreted as an interaction between ψ_L and ψ_R.

It is conventional to separate the space-time dependence

$$\psi = u e^{i(\vec{p}\cdot\vec{x} - p_o t)}$$

u satisfies the same momentum space equations we have been writing, since we were implicitly assuming that we were working with energy eigenstates.

Normalization: $\bar{u}u = 2m$

Antiparticles: the negative energy solutions of the free-particle equation correspond to positive energy antiparticles.
In a Feynman diagram an incoming (outgoing) particle can always be replaced by an outgoing (incoming) antiparticle.

Left-handed and Right-handed Fermions

Even when particles are not massless it is useful to separate the upper and lower parts of the wave function.

$$P_L = \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_R = \frac{1+\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are projection operators

$$\left.\begin{aligned} P_L^2 &= P_L \\ P_R^2 &= P_R \\ P_L + P_R &= 1 \\ P_L P_R &= 0 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} u_L &= P_L u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} U_R \\ U_L \end{pmatrix} = \begin{pmatrix} 0 \\ U_L \end{pmatrix} \\ u_R &= P_R u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} U_R \\ U_L \end{pmatrix} = \begin{pmatrix} U_R \\ 0 \end{pmatrix} \end{aligned}}$$

Helicity

The helicity of a massive fermion can be changed by a Lorentz transformation, so it is not a quantum number that can be used to label the system. Still left-handed and right-handed fermions are treated differently in the Standard Model.

Under parity the spin does not change, therefore there is a change in sign in the Dirac equation and the two solutions go into each other.

Thus if nature is invariant under the parity operation we expect both solutions to exist.

Neutrinos: only left-handed neutrinos exist.

Electrons: both left- and right-handed electrons exist, but they interact differently: e_L can interact directly with a neutrino, but e_R cannot.

Useful Relations

$$\bar{\psi}\gamma^{\mu}\psi = \bar{\psi}_L\gamma^{\mu}\psi_L + \bar{\psi}_R\gamma^{\mu}\psi_R$$

$$\bar{\psi}\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R$$

$$\boxed{\begin{aligned}\bar{\psi}_L\gamma^{\mu}\psi_L &= \frac{1}{4}\bar{\psi}\left(1+\gamma^5\right)\gamma^{\mu}\left(1-\gamma^5\right)\psi \\ &= \frac{1}{2}\bar{\psi}\gamma^{\mu}\left(1-\gamma^5\right)\psi\end{aligned}}$$

V-A One part transform as a vector and one as an axial vector.

$$\begin{aligned}\bar{\psi}_R\gamma^{\mu}\psi_R &= \frac{1}{4}\bar{\psi}\left(1-\gamma^5\right)\gamma^{\mu}\left(1+\gamma^5\right)\psi \\ &= \frac{1}{2}\bar{\psi}\gamma^{\mu}\left(1+\gamma^5\right)\psi\end{aligned}$$

The Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

- Lagrangian of a spin ½ fermion
- Used in the Euler-Lagrange equations it yields the Dirac equation.