4 Decay rates

In particle physics calculations, one is most often occupied with decay rates and cross sections: transitions from one state to another. Transitions can be calculated from Fermi’s Golden Rule:

$$\Gamma_{fi} = \frac{2\pi}{\hbar} |T_{fi}|^2 \rho(E_f)$$  \hspace{1cm} (1)

although from now on, let’s take natural units: \(\hbar = 1\)

- \(\Gamma_{fi}\) is the transition rate (number of transitions per unit time) from \(|i\rangle\) to \(|f\rangle\)
- \(T_{fi}\) is matrix element for \(|i\rangle \rightarrow |f\rangle\) \[|T_{fi}|^2 = |\langle \psi_f | V_{fi} | \psi_i \rangle|^2\]
- \(\rho(E_f)\): density of states available for energy \(E\); the so-called phase-space factor

The matrix element contains the information related to the dynamics of the interaction (exchanging force-carrier etc...).

To study \(T_{fi}\) this we need to be able to calculate the phase space factor \(\rho(E_f)\)

**Phase space (density of states)**

As the name suggests this is the number of states available per unit of energy in the final state. Let’s first imagine a cube of sides \(L\) containing one particle with quantised momentum (from \(p = \hbar k\) though take \(\hbar = 1\))

$$p_x = \frac{2\pi n_x}{L} \quad p_y = \frac{2\pi n_y}{L} \quad p_z = \frac{2\pi n_z}{L}$$  \hspace{1cm} (2)

Each \(\{n_x, n_y, n_z\}\) state resides in the elemental volume \((2\pi/L)^3 = (2\pi)^3/V\) in momentum space. If we normalise to one particle per [spatial] elemental volume, then the number of states available to that particle is:

$$dN_1 = \frac{\text{total phase space}}{\text{elemental volume}} = \frac{1}{(2\pi)^3 V} \int d^3p_x \int d^3p_y \int d^3p_z = \frac{1}{(2\pi)^3} \int d^3p$$

And for \(n\) particles:

$$dN_n = \frac{1}{(2\pi)^{3n}} \int \prod_{i=1}^n d^3p$$
And so the phase space: number states per unit energy:

\[
\rho(E_f) = \frac{dN_n}{dE} = \frac{1}{(2\pi)^3n} \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3p_i
\]

(3)

\[n - 1\] because total momentum conservation constrains the \(n\)th particle.

### Dirac \(\delta\)-functions

Dirac \(\delta\)-functions prove to be very useful tool in relativistic mechanics as they can be used to neatly encode conversation of energy and momentum. For example, in a decay of particle 1 to particles 2 and 3, one can write:

\[
\int \ldots \delta(E_1 - E_2 - E_3)dE \quad \text{and} \quad \int \ldots \delta(\vec{p}_1 - \vec{p}_2 - \vec{p}_3)d^3\vec{p}
\]

Important identities relating to these functions are:

\[
\int_{-\infty}^{+\infty} \delta(x - a) \, dx = 1
\]

\[
\int_{-\infty}^{+\infty} f(x)\delta(x - a) \, dx = f(a)
\]

\[
\delta(f(x)) = \left| \frac{df}{dx} \right|^{-1} \delta(x - x_0)
\]

(4)

Let’s re-express using Dirac \(\delta\)-functions for the momentum conversation:

\[
\vec{p}_n - \left( \vec{p} - \sum_{i=1}^{n-1} \vec{p}_i \right) = 0
\]

\[
\int d^3p_n \delta \left( \vec{p}_n - \left( \vec{p} - \sum_{i=1}^{n-1} \vec{p}_i \right) \right) = 1
\]

similarly:

\[
\int dE \delta \left( E - E_i \right) = 1
\]

\[
\rho(E_f) = \frac{1}{(2\pi)^3} \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3p_i \times \int d^3p_n \delta \left( \vec{p} - \sum_{i=1}^{n-1} \vec{p}_i \right) \times \int dE \delta \left( E - \sum_{i=1}^{n-1} E_i \right)
\]

\[
= \frac{1}{(2\pi)^3} \int d^3p_i \delta \left( \vec{p} - \sum_{i=1}^{n} \vec{p}_i \right) \delta \left( E - \sum_{i=1}^{n} E_i \right)
\]

(5)

All well and good but we still need an expression that is frame independent...

### Imposing Lorentz invariance

Lets go back to our particle, of energy \(E\) in volume \(V\).

\[
\int |\psi|^2 \, dV = 1
\]

This normalisation implies density = \(1/V\).

At relativistic speeds, Lorentz contraction in direction of motion will increase the particle density by \(\gamma/V\). This means the phase space derivation we have used so far is manifestly not Lorentz invariant.

So we requiring the wavefunction \(\psi\) to be defined with a normalisation \(\propto \sqrt{\gamma}\).
The usual convention is to normalize to 2E particles/unit volume

\[ \int |\psi|^2 \, dV = 2E \]

\[ \psi \rightarrow \sqrt{2E} \psi \]

So slightly redefining the matrix element:

\[ |M_{fi}|^2 = |T_{fi}|^2 \prod_{j=1}^{n_f} 2E_j \prod_{i=1}^{n_i} 2E_i \]

\( j \) represents a particle in the initial state (up to \( n_i \) particles). \( n \) continues to mean the number of particles in the final state.

So the Golden Rule becomes:

\[ \Gamma_{fi} = (2\pi)^4 \frac{1}{\prod_{j=1}^{n_f} 2E_j} \int |M_{fi}|^2 \prod_{i=1}^{n_i} \frac{d^3p_i}{(2\pi)^3(2E_i)} \delta \left[ p - \sum_{i=1}^{n_i} \vec{p}_i \right] \delta \left[ E - \sum_{i=1}^{n_i} E_i \right] \]

Note:

- \( |M_{if}|^2 \) used relativistically normalised wavefunctions. It is Lorentz invariant.
- \( \frac{d^3p_i}{(2\pi)^3(2E_i)} \) is the Lorentz invariant phase space for each final state particle.
- Energy and momentum are conserved by the \( \delta \)-functions.
- The \( (2\pi)^4 \) factor may seem a little arbitrary but it is a subtlety from a full treatment with QFT. The rule of thumb is a \( 2\pi \) is needed for every \( \delta \)-function employed.
- Eq. 6 is just a rearrangement of eq. 1 and eq. 5 but the integral is now frame independent.
- The transition rate is inversely proportional to energy of the initial state. This is exactly what we would expect in the case of a decaying particle whose decay rate is slowed by time dilation.

Exercise: show the Lorentz invariant phase space terms are indeed invariant under boost. Arbitrarily, choose boost in \( z \):

\[ \frac{dE}{dp_z} = \frac{d}{dp_z} \left( \sum_{j=xyz} p_j^2 + m^2 \right)^{\frac{1}{2}} = p_z \left( \sum_{j=xyz} p_j^2 + m^2 \right)^{-\frac{1}{2}} = \frac{p_z}{E} \]

\[ \frac{dp_{z'}}{dp_z} = \frac{\gamma \left( 1 - \beta \frac{dE}{dp_z} \right)}{E} = \frac{\gamma \left( 1 - \beta \frac{p_z}{E} \right)}{E} = \frac{\gamma \left( E - \beta p_z \right)}{E} \]

\[ \therefore \frac{dp_{z'}}{E} = \frac{dp_z}{E} \]

2-body decay

Because the integral is Lorentz invariant we are free to choose any frame. Choose CM frame: \( E_i = m_i, \ p_i = 0 \).

\[ \Gamma_{fi} = \frac{(2\pi)^4}{2m_i} \int |M_{fi}|^2 \frac{d^3p_1}{(2\pi)^3(2E_1)} \frac{d^3p_2}{(2\pi)^3(2E_2)} \delta \left[ \vec{p}_1 + \vec{p}_2 \right] \delta \left[ m_i - E_1 - E_2 \right] \]

\[ \Gamma_{fi} = \frac{1}{8m_i(2\pi)^2} \int |M_{fi}|^2 \frac{d^3p_1}{E_1E_2} \delta \left[ m_i - E_1 - E_2 \right] \]

\[ \Gamma_{fi} = \frac{1}{32\pi^2 m_i} \int |M_{fi}|^2 \frac{p_1^2 dp_1 d\Omega}{E_1E_2} \delta \left[ m_i - E_1 - E_2 \right] \]

(7)
as \( d^3 p = p_1^2 \, dp_1 \, \sin \theta \, d\theta \, d\phi \). Observing that \( E_2^2 = p_1^2 + m_2^2 \) because \( p_2 = -p_1 \), eq. 7 can be expressed as

\[
\Gamma_{fi} = \frac{1}{32\pi^2 m_i} \int |M_{fi}|^2 g(p_1) \delta(f(p_1)) \, dp_1 \, d\Omega
\]

where

\[
g(p_1) = p_1^2/(E_1 E_2) = p_1^2 \sqrt{(m_1^2 + p_1^2)/(m_1^2 + p_1^2)}
\]

\[
f(p_1) = m_i - \sqrt{(m_1^2 + p_1^2) - \sqrt{(m_1^2 + p_1^2)}}
\]

Using eq. 4, \( g(p_1) \) and \( \delta(f(p_1)) \) can be integrated:

\[
\int g(p_1) \delta(f(p_1)) \, dp_1 = \frac{1}{|df/dp_1|_{p^*}} \int g(p_1) \delta(p_1 - p^*) \, dp_1 = \frac{g(p^*)}{|df/dp_1|_{p^*}}
\]

where \( p^* \) is the value of \( p_1 \) that satisfies the original \( \delta \)-function; i.e. momentum conservation. Finally,

\[
\frac{df}{dp_1} = - \frac{p_1}{\sqrt{m_1^2 + p_1^2}} - \frac{p_1}{\sqrt{m_2^2 + p_1^2}} = - \frac{p_1}{E_1} - \frac{p_1}{E_2} = - \frac{E_1 + E_2}{E_1 E_2}
\]

Finally giving:

\[
\Gamma_{fi} = \frac{1}{32\pi^2 m_i} \int |M_{fi}|^2 \frac{E_1 E_2}{p^* (E_1 + E_2)} \frac{p^2}{E_1 E_2} \, d\Omega
\]

\[
\Gamma_{fi} = \frac{1}{32\pi^2 m_i} \int |M_{fi}|^2 \frac{p^*}{E_1 + E_2} \, d\Omega
\]

and with \( E_1 + E_2 = m_i \)

\[
\frac{1}{\tau} = \Gamma_{fi} = \frac{|p^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, d\Omega
\]

This is valid for all two-body decays. the momentum in the CM frame, \( p^* \) can be obtained from \( f(p_1) = 0 \)

\[
m_i = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}
\]

\[
p^* = \frac{1}{2m_i} \sqrt{[m_i^2 - (m_1 + m_2)^2][m_i^2 - (m_1 - m_2)^2]}
\]

**Phase space of 3-body decays**

Write the Lorentz sum of \( m_1 \) and \( m_2 \) explicitly in terms of \( E_3 \) and \( p_3 \):

\[
M_{12}^2 = (E_1 + E_2)^2 - (p_1 + p_2)^2 = (M - E_3)^2 - p_3^2
\]

And identify \( m_3^2 \):

\[
M_{12}^2 = M^2 - 2ME_3 + E_3^2 - p_3^2
\]

\[
= M^2 + m_3^2 - 2ME_3
\]

One can show (exercise!) that:

\[
dp(E_f) \propto dE_1 
\]

\[
\delta \propto (E_1 + E_2 + E_3 - M)
\]

Finally from the differential of eq. 8: \( d(M_{12}^2) \propto dE_3 \) etc.:

\[
dR_3 \propto d(M_{12}^2) \, d(M_{13}^2)
\]

Phase space available is uniform in the 2D distribution of \( M_{12}^2 \) v.s. \( M_{13}^2 \). This means the dynamical structure is easy to pick out. Such distributions are called Dalitz plots.