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Four-dimensional Treatment of Linear Acoustic Fields and Radiation Pressure

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Summary
A general expression of acoustic radiation pressure is here derived on the basis of the linear theory of classical fields. Following this theory, the acoustic energy, the sound intensity and the sound momentum density are introduced, together with the 3×3 wave-momentum flux density tensor, as components of a 4×4 acoustic energy-momentum tensor in a unified space-time approach, formally similar to the relativistic formulation of electromagnetism. The related conservation laws are then expressed by the condition of vanishing 4-divergence of this tensor, showing in particular that the so-called radiation pressure is nothing but a consequence of the momentum conservation law for the acoustic field. As an application, the radiation pressure is computed explicitly in two cases: a plane wave reflected on a flat wall and the field in the interior of an open organ pipe. In the latter case, indirect measurements of the radiation pressure have been also performed by an intensimetric technique, allowing to determine the complex reflection amplitude at the pipe's end. Finally, as an appendix of the paper, the angular momentum conservation and the analogy between the acoustic and electromagnetic radiation pressure are analyzed to some extent.
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1. Introduction

Following our previous works, mainly concerned with the rigorous definition of time averaged energetic properties of general linear acoustic fields (e.g. [1], [2], [3], [4]), the first aim of this paper is to cast a link between the energy and momentum density concepts from the point of view of their conservation laws formulated in the so-called acoustic space-time, to be defined in subsection 2.2. The accomplishment of this task has allowed us to give a contribution to the study of a classical research subject: the acoustic radiation pressure, a physical quantity that in our view is simply a consequence of the momentum conservation law for acoustic fields. Therefore, this quantity plays a role in all fields of acoustics, including the audible linear domain, being complementary and quite similar to that played by intensity for energy conservation.

The second aim of this paper is to express the radiation pressure as a time-dependent quantity in terms of the solution of the wave equation with appropriate boundary conditions, without relying necessarily on the two historical definitions of radiation pressure, due to Rayleigh and to Langevin, which often predominate in the acoustical literature [5]. Furthermore, our approach can be extended in principle to nonlinear acoustics, since it is based on the general physical principles of field theory.

The theoretical definition of acoustic radiation pressure given here follows the development of the analogous quantity for the electromagnetic field, since it is based on the 3×3 wave-momentum flux density tensor as a part of the 4×4 energy-momentum tensor. In the electromagnetic case, the momentum flux density tensor was introduced by James C. Maxwell in his famous *Treatise on Electricity and Magnetism,* published in 1873 [6], and was there subdivided in two parts, treated separately: the electric part, called the *electrostatic* stress, and the magnetic one, or *electrokinetic* stress (see Art.s 105-111 and 639-646 of the Treatise). In this way, Maxwell obtained an expression of the force which arises in the field itself due to the presence of electromagnetic waves, the so-called "radiation pressure" (Art.s 792-793). According to this conception, radiation pressure turns out to be a non-linear effect, but just in the sense that it is a second-order quantity derived from a linear wave equation.

In the linear acoustics context this approach allowed us to obtain a time-dependent form of the same quantity, which has been named *acoustic radiation pressure* thanks to the electromagnetic analogy. One interesting point is that the present treatment is formally in agreement with the one given by Beissner [7], which follows rigorously from the fluid mechanical theory based on the momentum

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flux density as given for instance in [8]. In fact, the two approaches can be considered equivalent for they lead to the same time-average expression of the tensor, whose surface integral is properly called "radiation force" by Beissner, due to its vectorial nature.

Besides classical references on radiation pressure where the subject is mainly considered and usually presented as a linear or non-linear effect, arising anyway from a particular choice of the field boundary conditions ([9], [10], [11], [12], [13]), the treatment of the acoustic radiation pressure from the point of view of the general field theory most similar to that presented here is due to Morse and Ingard [14]: there, the Lagrangian density, the acoustic stress tensor and other quantities and equations related to linear acoustic fields are introduced regardless of any boundary condition. Moreover, the validity of the expressions reported for radiation pressure by Morse is limited to the case of plane waves. Radiation pressure is treated in the same framework also by Brillouin [15], but only in the case of elastic solid bodies. Anyway, unlike the present paper, both the above treatments follow the standard 3-dimensional approach.

As regards the content of the paper, the Lagrangian theory of acoustic fields and the conservation laws of energy and momentum are treated in Sect. 2, the general properties of acoustic radiation pressure are described in Sect. 3. Section 4 is then concerned with radiation pressure produced by a plane wave incident on a flat wall: in particular, the dependence on boundary conditions is discussed. Finally, Sect. 5 treats the case of the acoustic field inside an organ pipe: a simple theoretical model for describing the inner field and an indirect method for measuring radiation pressure by means of the intensimetric technique are presented. In the same section, it is shown how this method can be practically employed for determining the impedance of the upper end of the pipe.

For sake of completeness, two appendices at the end of the paper briefly present a derivation of the conservation law of the acoustic angular momentum and a short comparison of the acoustics radiation pressure with the electromagnetic case.

The general treatment of this subject requires a certain knowledge of the foundations of the classical theory of fields, as it can be found for instance in references [16], [17]; nevertheless, some efforts will be spent for presenting the subject in a self-contained form in order to keep the references to other works to a minimum.

2. Lagrangian theory of acoustic fields

2.1. Review of the linear equations of acoustics

As a starting point of the whole discussion, it is convenient to review the basic principles and assumptions which the linear theory of acoustic fields is based upon. We consider the case of an isentropic motion taking place in a perfect gas. The three fundamental equations are:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{2}$$

$$p = \mathrm{const}\cdot\rho^{\gamma}. \tag{3}$$

Equation (1) is the equation of motion (Euler equation), equation (2) is the mass conservation equation and equation (3) is the equation for an adiabatic trasformation. From the assumption that the acoustic perturbations ρ', p' are small compared to the equilibrium values of the unperturbed fluid (ρ_e, p_e), which is also considered uniform and still (i.e. $\nabla\rho_\mathrm{e} = \mathbf{0}$, $\nabla p_\mathrm{e} = \mathbf{0}$, $\mathbf{v}_\mathrm{e} = \mathbf{0}$), the equations above can be approximated to the first order as

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{1}{\rho_\mathrm{e}}\nabla p', \tag{4}$$

$$\frac{\partial \rho'}{\partial t} + \rho_\mathrm{e}\nabla\cdot\mathbf{v}' = 0, \tag{5}$$

$$\frac{p'}{\rho'} = \gamma\frac{p_\mathrm{e}}{\rho_\mathrm{e}} =: c^2. \tag{6}$$

The d'Alembert equation for the acoustic variables can then be derived (see e.g. [14] p. 243)

$$\Delta p' - \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = 0, \tag{7}$$

$$\Delta \rho' - \frac{1}{c^2}\frac{\partial^2 \rho'}{\partial t^2} = 0, \tag{8}$$

$$\Delta \mathbf{v}' - \frac{1}{c^2}\frac{\partial^2 \mathbf{v}'}{\partial t^2} = 0. \tag{9}$$

where the symbol Δ stands for the Laplacian operator ∇^2 i.e. the divergence of a gradient.

In the context of linear acoustics a real scalar field $\phi(\mathbf{x}, t)$ (the kinetic potential) can be introduced in such a way that the perturbations of pressure, density and velocity are given by:

$$p' = -\rho_\mathrm{e}\frac{\partial\phi}{\partial t}, \qquad \rho' = -\frac{\rho_\mathrm{e}}{c^2}\frac{\partial\phi}{\partial t}, \qquad \mathbf{v}' = \nabla\phi. \tag{10}$$

The kinetic potential too satisfies the d'Alembert equation

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{11}$$

The energy and momentum of the acoustic field are defined by isolating from the corresponding quantities related to the fluid in motion the terms due to the acoustic wave only. The acoustic energy density $\mathcal{W}'$ is calculated from the quantity $\mathcal{W} = \rho\varepsilon + \rho\mathbf{v}^2/2$, where ε is the specific internal energy (energy per unit mass) and ρ, $\mathbf{v}$ are the variables describing the fluid motion. Expanding $\mathcal{W}$ in a Taylor's series in the acoustic variables and stopping the process to the second-order approximants one obtains

$$\mathcal{W}' := \frac{1}{2}\rho_\mathrm{e}\left(\frac{p'^2}{z^2} + \mathbf{v}'^2\right), \tag{12}$$

where $z := \rho_e c$ is the characteristic impedance of the medium. The energy density $\mathcal{W}'$ is made of a compressional part $\mathcal{U}'$ and a kinetic part $\mathcal{K}'$:

$$\mathcal{U}' := \frac{1}{2}\rho_e \frac{p'^2}{z^2}, \quad \mathcal{K}' := \frac{1}{2}\rho_e \mathbf{v}'^2. \tag{13}$$

It will be shown below that the energy density of equation (12) is linked to the acoustic field in the linear approximation, since the d'Alembert wave equation is deducible from the Lagrangian density $\mathcal{L}' = \mathcal{K}' - \mathcal{U}'$ through the variational principle.

The wave-momentum is obtained in an analogous way from the fluid momentum density $\mathbf{q} = \rho\mathbf{v} = (\rho_e + \rho')\,\mathbf{v}'$. Integrating $\mathbf{q}$ on a sufficiently large space region, it can be seen that the total instantaneous momentum of the fluid reduces to the integral of the term $\mathbf{q}' = \rho'\mathbf{v}'$. As a consequence, $\mathbf{q}'$ is interpreted as the sound momentum density.

2.2. Geometry of acoustic space-time

We note that the d'Alembert equation (11) can be considered as a description of a real, scalar, massless field in the language of relativistic quantum field theory. Therefore, it is possible to describe acoustic phenomena within the Lagrangian framework treated in standard textbooks [16], [17]: this can be achieved simply by replacing the speed of light with the speed of sound. Of course, the acoustic case presents only a formal analogy with relativity, since the Lorentz transformations obtained in this way, to be referred hereafter as *acoustic Lorentz transformations*, still form an invariance group of the wave equation, but do not connect inertial reference frames. The physical meaning of such transformations is related to the study of the radiation from a point source, moving with constant speed (Ref. [14], Sect. 11.2). From another point of view, acoustic Lorentz transformations connect all space-time coordinates, which describe sound propagation with the same speed c and therefore leave invariant the wave equation.

Since the common notation of relativistic field theory is not usual in acoustics, a short account of it is given in the present and in the following subsections. Let us first introduce the ordinary *euclidean 3-dimensional space* $\mathsf{E}_3 = (\mathsf{R}^3, d)$, made of the real 3-dimensional vector space R^3 with the euclidean metric d. Any vector $\mathbf{x}$ in R^3 can be represented in terms of any set of three linearly independent vectors $\{\mathbf{e}_i\}_{i=1,2,3}$ and components x^i as

$$\mathbf{x} = \sum_{i=1}^{3} x^i \mathbf{e}_i =: x^i \mathbf{e}_i. \tag{14}$$

Here, we have introduced the *summation convention in* E_3, meaning that a pair of equal upper and lower Latin indices denotes summation over all the values from 1 to 3. The euclidean metric is defined by means of the ordinary scalar product

$$\mathbf{x}\cdot\mathbf{y} = \mathbf{e}_i x^i \cdot \mathbf{e}_j y^j = e_{ij} x^i y^j \tag{15}$$

as $d(\mathbf{x},\mathbf{y}) = \sqrt{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})} = |\mathbf{x}-\mathbf{y}|$, (16)

where $e_{ij} := \mathbf{e}_i \cdot \mathbf{e}_j$ are the components of the *euclidean metric tensor*

$$\mathsf{e}' = e_{ij}\mathbf{e}^i \otimes \mathbf{e}^j, \tag{17}$$

where $\{\mathbf{e}^i\}_{i=1,2,3}$ is the dual basis of $\{\mathbf{e}_i\}$, defined by

$$(\mathbf{e}^i, \mathbf{e}_j) = \delta^i_j = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \tag{18}$$

in terms of the duality mapping $(,)$ and $\otimes$ denotes the tensor (or diadic) product. The matrix $\{e_{ij}\}$ is non-singular, positive-definite and in Cartesian coordinates is represented by the unit matrix

$$e_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases}. \tag{19}$$

Vectors $\mathbf{x}'$ of the dual space E_3' are represented by

$$\mathbf{x}' = x_i \mathbf{e}^i, \tag{20}$$

where the components x_i are connected to x^j by $x_i = e_{ij}x^j$, so that in Cartesian coordinates $x_i = x^i$. The components x_i transform under linear transformations according to the same law as the basis vectors $\mathbf{e}_i$ and, therefore, are called *covariant components*, in order to distinguish them from the *contravariant components* x^i, which transform in an opposite way compared with $\mathbf{e}_i$. In this way, an invariant vector $x^i\mathbf{e}_i$ is obtained. The tensor e' belongs to the space $\mathsf{E}_3' \otimes \mathsf{E}_3'$.

To describe an event which happens at the point $\mathbf{x}$ and at time t, it is convenient to introduce a fourth dimension, labelled by 0, and set $x^0 = ct$, where c is the speed of sound. Thus, the coordinate x^0 represents the distance covered by sound in time t. The 4-dimensional real space R^4, whose vectors x have components x^0, x^1, x^2, x^3 is what we mean by *acoustic space-time*. Let us now represent x in terms of the basis vectors $\{g_\lambda\}_{\lambda=0,1,2,3}$ as

$$x = \sum_{\lambda=0}^{3} x^\lambda g_\lambda =: x^\lambda g_\lambda. \tag{21}$$

Here, the definition introduces the *summation convention in the acoustic space-time*, meaning that a pair of equal upper and lower Greek indices denotes summation on all values from 0 to 3. In order to leave the wave equation invariant under acoustic Lorentz transformations, we introduce in R^4 a *pseudo-euclidean* metric m, thus obtaining the *acoustic Minkowski space* $\mathsf{M}_4 = (\mathsf{R}^4, m)$. The pseudo-euclidean metric is defined by means of the non-positive-definite scalar product

$$x \odot y = x^\lambda g_\lambda \odot y^\mu g_\mu = g_{\lambda\mu} x^\lambda y^\mu \tag{22}$$

as

$$\begin{aligned} [m(x,y)]^2 &= (x-y)\odot(x-y) \\ &= g_{\lambda\mu}(x^\lambda - y^\lambda)(x^\mu - y^\mu), \end{aligned} \tag{23}$$

where $g_{\lambda\mu} := g_\lambda \odot g_\mu$ are the components of the *pseudo-euclidean metric tensor*

$$g = g_{\lambda\mu}\left(g^\lambda \otimes g^\mu\right) \tag{24}$$

and g^λ are the dual basis vectors of g_λ

$$\left(g^\lambda, g_\mu\right) = \delta^\lambda_\mu = \begin{cases} 1, \text{if } \lambda = \mu, \\ 0, \text{if } \lambda \neq \mu, \end{cases}$$
$$(\lambda, \mu = 0,1,2,3), \tag{25}$$

with the standard matrix representation of the metric tensor

$$g_{\lambda\mu} = \begin{cases} 1 \text{ , if } \lambda = \mu = 0 \\ 0 \text{ , if } \lambda \neq \mu \\ -1 \text{ , if } \lambda = \mu = 1,2,3 \end{cases}. \tag{26}$$

The matrix $\{g_{\lambda\mu}\}$ is still non-singular, but not positive-definite any more. The components $g^{\mu\lambda}$ of the inverse metric tensor are defined by

$$g^{\mu\lambda} g_{\nu\mu} = \delta^\lambda_\nu \tag{27}$$

and therefore $g^{\mu\lambda} = g_{\mu\lambda}$. Let us denote by x' the dual vector of x, expressed by

$$x' = x_\lambda g^\lambda; \tag{28}$$

its components are given by

$$x_\lambda = g_{\lambda\mu} x^\mu, \quad x_0 = x^0, \quad x_i = -x^i,$$
$$(i = 1,2,3). \tag{29}$$

Differentiation operators with respect to coordinates of spaces E_3 and M_4 are denoted by

$$\nabla = \mathbf{e}^k \partial_k, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_0 = \frac{1}{c}\frac{\partial}{\partial t},$$
$$(k = 1,2,3), \quad (\mu = 0,1,2,3). \tag{30}$$

2.3. The wave equation in four-dimensional form

Coming back to the acoustic quantities, the pressure and velocity perturbations may then be written as:

$$p = -z\partial_0\phi, \qquad \mathbf{v} = v_k \mathbf{e}^k = \mathbf{e}^k \partial_k \phi = \nabla\phi, \tag{31}$$

(from now on the primes on the acoustic quantities will be omitted for simplicity, since the context is now clearly that of linear acoustics).

The kinetic, potential and total energy densities, $\mathcal{K}$, $\mathcal{U}$, $\mathcal{W}$, of an acoustic field in the lowest order approximation are given by the second order quantities (see equation 12)

$$\mathcal{K} = \frac{1}{2}\rho_\mathrm{e} e^{ij} \left(\partial_i \phi\right) \left(\partial_j \phi\right) = \frac{1}{2}\rho_\mathrm{e} \left|\nabla\phi\right|^2, \tag{32}$$

$$\mathcal{U} = \frac{1}{2}\rho_\mathrm{e} \left(\partial_0\phi\right)^2 = \frac{1}{2}\rho_\mathrm{e} \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2, \tag{33}$$

$$\mathcal{W} = \mathcal{K} + \mathcal{U} = \frac{1}{2}\rho_\mathrm{e}\left[e^{ij}\left(\partial_i\phi\right)\left(\partial_j\phi\right) + \left(\partial_0\phi\right)^2\right]. \tag{34}$$

With the above positions, the Lagrangian density $\mathcal{L} := \mathcal{K} - \mathcal{U}$ can be written as an expression which is invariant under acoustic Lorentz transformations:

$$\mathcal{L} = -\frac{1}{2}\rho_\mathrm{e} g^{\mu\nu} \left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right). \tag{35}$$

The wave equation may be obtained from the Lagrangian density by means of a well-known variational principle [16],[17]: this leads to the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\phi_\mu} = 0, \quad \phi_\mu := \partial_\mu\phi, \tag{36}$$

which in our case takes the form of the above-mentioned d'Alembert equation

$$\Box\phi = 0, \quad \left(\Box := g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right). \tag{37}$$

The generalized momentum is a vector $P \in \mathsf{M}_4$, whose components are defined as

$$P^\mu := \frac{\partial\mathcal{L}}{\partial\phi_\mu} = -\rho_\mathrm{e} g^{\mu\nu}\phi_\nu = -\rho_\mathrm{e}\phi^\mu. \tag{38}$$

Its component P^0 is equal to the acoustic pressure p divided by c

$$P^0 = -\rho_\mathrm{e} g^{00}\phi_0 = -\frac{\rho_\mathrm{e}}{c}\frac{\partial\phi}{\partial t} = \frac{p}{c}, \tag{39}$$

while the other components are equal to the momentum density of the gas:

$$P^i = -\rho_\mathrm{e} g^{i\nu}\phi_\nu = \rho_\mathrm{e}\phi_i = \rho_\mathrm{e}\frac{\partial\phi}{\partial x^i} = \rho_\mathrm{e} v_i$$
$$(i = 1,2,3). \tag{40}$$

It is then easy to see that the wave equation (equation 37) coincides with the condition of null divergence of the generalized momentum in M_4:

$$\partial_\mu P^\mu = 0. \tag{41}$$

We want now to illustrate the physical meaning of the squared length of a vector $x \in \mathsf{M}_4$:

$$x \odot x = g_{\lambda\mu} x^\lambda x^\mu = g^{\lambda\mu} x_\lambda x_\mu = c^2t^2 - |\mathbf{x}|^2. \tag{42}$$

The set of all vectors of vanishing length is the four-dimensional cone of equation $c^2t^2 = |\mathbf{x}|^2$, analogous to the light-cone of the relativistic theory, with axis in the direction of x^0-axis and vertex in $x = 0$: this is a characteristic surface of the wave equation. The transformations leaving invariant the scalar product of equation (42) and the characteristic surfaces of the wave equation are just the acoustic Lorentz transformations. The physical meaning of the cone, which in our case should be called *sound-cone*, is related to the causality principle: only all points inside the *future sound-cone*, i.e. points such that $c^2t^2 - |\mathbf{x}|^2 \geq 0$, $t \geq 0$, can be the support of an acoustic field (*effect*), produced by a source (*cause*) placed at the point $\mathbf{x} = \mathbf{0}$ and switched on at time $t = 0$.

2.4. The four-dimensional energy-momentum tensor and conservation laws

A well known result of field theory is the following. If the Lagrangian density is invariant under coordinate or field transformations of a certain group, there exists a conserved quantity, whose conservation law is expressed as the vanishing of the divergence of a certain tensor (Emmy Nöther's theorem [16]). The Lagrangian of equation (35) and the wave equation (equation 37) are invariant under both acoustic Lorentz transformations (see equation A1 in Appendix) and translations in M_4

$$\tilde{x}^\mu = x^\mu + a^\mu, \quad (\mu = 0,1,2,3). \tag{43}$$

The relation of the first invariance property to angular momentum conservation will be treated in Appendix, with the purpose of displaying the generality of the field theoretical method. The conserved quantities corresponding to the second invariance property are the acoustic energy and momentum; the tensor, whose vanishing divergence represents the related conservation laws, is the *energy-momentum tensor*

$$T := [\rho_e \phi^\mu \phi^\nu + g^{\mu\nu}\mathcal{L}]\, g_\mu \otimes g_\nu \quad \in \ \mathsf{M}_4 \otimes \mathsf{M}_4. \tag{44}$$

The component T^{00} is equal to the acoustic energy density $\mathcal{W}$:

$$\begin{aligned} T^{00} &= \rho_e \phi^0\phi^0 + \mathcal{L} \\ &= \frac{1}{2}\rho_e\left(\phi_0\phi_0 + \phi_1\phi_1 + \phi_2\phi_2 + \phi_3\phi_3\right) = \mathcal{W}. \end{aligned} \tag{45}$$

The other components are

$$\begin{aligned} T^{0i} &= T^{i0} = \rho_e\phi^0\phi^i = \frac{p\phi_i}{c} = \frac{pv_i}{c} = \frac{j_i}{c}, \\ T^{ik} &= T^{ki} = \rho_e\phi^i\phi^k + g^{ik}\mathcal{L} \\ &= \rho_e v_i v_k + \frac{1}{2}\rho_e\left(\frac{p^2}{z^2} - \mathbf{v}^2\right)e_{ik}, \quad (i,k = 1,2,3), \end{aligned} \tag{46}$$

where the E_3-vector

$$\mathbf{j} := cT^{0k}\mathbf{e}_k = p\mathbf{v} = c^2\mathbf{q} \tag{47}$$

is the *acoustic energy-flux density* (instantaneous acoustic intensity) and the E_3-vector $\mathbf{q}$, defined as

$$\mathbf{q} := \frac{p}{c^2}\mathbf{v} = \rho\mathbf{v} \tag{48}$$

is the acoustic momentum density.

The tensor T can be represented by the 4×4 matrix

$$T = \begin{pmatrix} \mathcal{W} & \mathbf{t} \\ \mathbf{t} & \mathsf{T} \end{pmatrix}, \tag{49}$$

where the components of the vector $\mathbf{t} = \mathbf{j}/c = c\mathbf{q} \in \mathsf{E}_3$ occupy the last three places of the first row and of the first column; T is the tensor of space $\mathsf{E}_3 \otimes \mathsf{E}_3$, represented by

$$\begin{aligned} \mathsf{T} &= \rho_e \mathbf{v}\otimes\mathbf{v} - \mathcal{L}\mathsf{e} = \rho_e\left[\mathbf{v}\otimes\mathbf{v} + \frac{1}{2}\left(p^2/z^2 - \mathbf{v}^2\right)\mathsf{e}\right] \\ &= \frac{\rho_e}{2}\begin{pmatrix} v_1^2 - v_2^2 - v_3^2 + p^2/z^2 & 2v_1v_2 & 2v_1v_3 \\ 2v_2v_1 & v_2^2 - v_3^2 - v_1^2 + p^2/z^2 & 2v_2v_3 \\ 2v_3v_1 & 2v_3v_2 & v_3^2 - v_1^2 - v_2^2 + p^2/z^2 \end{pmatrix} \end{aligned} \tag{50}$$

and $\mathsf{e} = e^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. Note that Morse and Ingard denote the tensor T by $-\mathsf{W}$ and call it *wave-stress tensor* [14].

Let us now examine some properties of the tensor T. First of all, its trace is equal to twice the Lagrangian density:

$$\mathrm{Tr}\,(T) := T^\mu_\mu = \rho_e \phi^\mu\phi_\mu + g^\mu_\mu\mathcal{L} = 2\mathcal{L}. \tag{51}$$

We now prove that the divergence of T vanishes identically. Putting $\phi_{\mu\nu} := \partial_\mu\partial_\nu\phi$, we find:

$$\begin{aligned} \partial_\nu T^{\mu\nu} &= \rho_e\partial_\nu\left[\phi^\mu\phi^\nu + g^{\mu\nu}\mathcal{L}\right] \\ &= \rho_e\left[\phi^\mu_\nu\phi^\nu - \frac{1}{2}\left(\phi^\nu\phi^\mu_\nu + \phi^\mu_\nu\phi^\nu\right)\right] \equiv 0. \end{aligned} \tag{52}$$

The component $\mu = 0$ of equation (52) can be written in the form

$$\begin{aligned} \partial_\nu T^{0\nu} &= \partial_0 T^{00} + \partial_1 T^{01} + \partial_2 T^{02} + \partial_3 T^{03} \\ &= \frac{1}{c}\left(\frac{\partial\mathcal{W}}{\partial t} + \nabla\cdot\mathbf{j}\right) = 0, \end{aligned} \tag{53}$$

and therefore represents the conservation law of acoustic energy.

The components $\mu = i = 1,2,3$ of equation (52) can be written in vector form as

$$\frac{\partial\mathbf{q}(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathsf{T}(\mathbf{x},t) = \mathbf{0}, \quad \nabla\cdot\mathsf{T} := \mathbf{e}_i\partial_j T^{ij}, \tag{54}$$

so that this equation represents the acoustic momentum conservation law, if the vector $-\nabla\cdot\mathsf{T}$ is interpreted as a force density. The integral form of equation (54) can be found by integrating this equation on any fixed volume V:

$$\int_V \left[\frac{\partial\mathbf{q}(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathsf{T}(\mathbf{x},t)\right]\mathrm{d}^3\mathbf{x} = \mathbf{0}. \tag{55}$$

Denoting by S the surface enclosing the volume V, by $\mathbf{n}$ its normal unit vector pointing outwards of V and putting $\mathbf{f}(\mathbf{x},t) := -\nabla\cdot\mathsf{T}(\mathbf{x},t)$, $\mathbf{s}(\mathbf{x},t) := -\mathsf{T}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})$, equation (55) can be rewritten in either form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{q}(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x} = \int_V \mathbf{f}(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x}, \tag{56}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{q}(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x} = \int_S \mathbf{s}(\mathbf{x},t)\,\mathrm{d}^2\mathbf{x}, \tag{57}$$

where we have used the divergence theorem to obtain equation (57). In equations (56), (57) the left-hand sides are the time derivative of the momentum produced by the acoustic wave in the volume V, so that the right-hand sides represent the force of the radiation field, expressed by means of a volume force in the first equation and a surface force in the second one. Thus, the vectors $\mathbf{f}$ and $\mathbf{s}$ are respectively the *volume force density* and the *surface force density* of the acoustic field, the latter representing the instantaneous *acoustic radiation pressure*. Note that the time average value of $\mathbf{s}$ integrated over a surface of finite area has been called *radiation force* by Beissner [7].

Since $\mathbf{s} = -\mathsf{T} \cdot \mathbf{n}$, we can state that the variation per unit time of the wave momentum in volume V is equal to the flux of the tensor $-\mathsf{T}$ through the surface S. Therefore, T represents the *wave-momentum flux density* entering V and so will be called in the following. Keeping in mind equation (50), the identities of vectorial analysis

$$\begin{aligned} \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) &\equiv (\nabla \cdot \mathbf{v})\, \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v}, \\ (\mathbf{v} \cdot \nabla)\, \mathbf{v} &\equiv \frac{1}{2} \nabla \left(\mathbf{v}^2\right) - \mathbf{v} \wedge (\nabla \wedge \mathbf{v}), \\ \nabla \cdot \left(p^2 \mathsf{e}\right) &\equiv \nabla \left(p^2\right) \end{aligned} \tag{58}$$

($\wedge$ denotes the vector product) and the condition of irrotational motion $\nabla \wedge \mathbf{v} = \mathbf{0}$, we find the following expression of the force density

$$\mathbf{f} = -\rho_{\mathrm{e}} (\nabla \cdot \mathbf{v})\, \mathbf{v} - \frac{1}{2zc} \nabla \left(p^2\right), \tag{59}$$

whose second term is related to the reactive intensity [1].

2.5. Conservation laws derived from linear acoustic theory

We now want to check that the conservation equations obtained above as a consequence of the invariance properties of the Lagrangian density can also be directly calculated from the definition of sound energy in linear acoustics (see subsection 2.1).

The first relation to consider is the energy conservation: this may be found from the time derivative of the energy density $\mathcal{W}$ expressed by equation (12):

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial t} &= \rho_{\mathrm{e}} \left(\frac{1}{z^2} p \frac{\partial p}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \right) \\ &= \rho_{\mathrm{e}} \left(\frac{c^2}{z^2} p \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \right), \end{aligned} \tag{60}$$

where in the right-hand term the substitution $\rho = p/c^2$ has been made, according to the linear approximation of equation of state (equation 6). Taking now into account the first-order versions for the mass conservation equation (equation 5) and the Euler equation (equation 4) one obtains

$$\frac{\partial \mathcal{W}}{\partial t} = -p \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla p = -\nabla \cdot (p \mathbf{v}), \tag{61}$$

from which it can be inferred that the quantity $\mathbf{j} = p\mathbf{v}$ expresses the energy flux density (instantaneous intensity) for the acoustic field described by the linear wave equation (equation 11).

As regards the sound momentum density $\mathbf{q} = \rho \mathbf{v}$, the time derivative is given by

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial t} &= \frac{\partial (\rho \mathbf{v})}{\partial t} = \frac{\partial \rho}{\partial t} \mathbf{v} + \rho \frac{\partial \mathbf{v}}{\partial t} \\ &= -\mathbf{v} \nabla \cdot (\rho_{\mathrm{e}} \mathbf{v}) - \rho \left(\frac{\nabla p}{\rho_{\mathrm{e}}} \right), \end{aligned} \tag{62}$$

where again equations (5) and (4) have been used. Thanks to equation (58) it is then easily seen that the two terms of the last expression can be written as follows

$$\rho_{\mathrm{e}} \mathbf{v} \nabla \cdot \mathbf{v} = \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \frac{1}{2} \nabla \cdot \left(\mathbf{v}^2 \mathsf{e}\right), \tag{63}$$

$$\begin{aligned} \rho \left(\frac{\nabla p}{\rho_{\mathrm{e}}} \right) &= \frac{p}{\rho_{\mathrm{e}} c^2} \nabla p = \frac{1}{2 \rho_{\mathrm{e}} c^2} \nabla \left(p^2\right) \\ &= \frac{1}{2 \rho_{\mathrm{e}} c^2} \nabla \cdot \left(p^2 \mathsf{e}\right). \end{aligned} \tag{64}$$

We then obtain the relationship

$$\frac{\partial \mathbf{q}}{\partial t} = -\nabla \cdot \left\{ \rho_{\mathrm{e}} \left[\mathbf{v} \otimes \mathbf{v} + \frac{1}{2} \left(\frac{p^2}{z^2} - \mathbf{v}^2 \right) \mathsf{e} \right] \right\}, \tag{65}$$

which confirms that $\mathsf{T} = \rho_{\mathrm{e}} \mathbf{v} \otimes \mathbf{v} - \mathcal{L} \mathsf{e}$ (see equation 50) represents the wave-momentum flux density for the linear acoustic field.

3. Acoustic radiation pressure

We have seen that the symmetric tensor T (equation 50) is the wave momentum flux density and therefore the quantity

$$\mathbf{s} = -\rho_{\mathrm{e}} \left[v_{\mathrm{n}} \mathbf{v} + \frac{1}{2} \left(\frac{p^2}{z^2} - \mathbf{v}^2 \right) \mathbf{n} \right] \tag{66}$$

is the acoustic radiation pressure, that is the force density exerted on an arbitrary ideal surface (not necessarily a physical target!) having normal $\mathbf{n}$ within the field itself. Furthermore, since the wave momentum flux density at the surface of a possible wall bounding the sound field will depend on the boundary conditions there, it is clear that the radiation pressure in turn will also depend on such conditions: in particular, in section 4 it will be shown that it is maximum when the wall is perfectly reflecting.

Let us now consider the directional properties of radiation pressure. We recall that any tensor $\mathsf{W} = W^{ij} \mathbf{e}_i \otimes \mathbf{e}_j \in \mathsf{E}_3 \otimes \mathsf{E}_3$ can be decomposed in a unique way into the sum of an *isotropic* one $\mathsf{W}^{(0)} = (1/3)\mathrm{Tr}(\mathsf{W})\mathsf{e}$ (the components of an isotropic tensor, by definition, are invariant under rotations), an *antisymmetric* one $\mathsf{W}^{(1)} = (1/2)(W^{ij} - W^{ji})\mathbf{e}_i \otimes \mathbf{e}_j$ and a *symmetric*, *traceless* one $\mathsf{W}^{(2)} = (1/2)(W^{ij} + W^{ji})\mathbf{e}_i \otimes \mathbf{e}_j - (1/3)\mathrm{Tr}(\mathsf{W})\mathsf{e}$: $\mathsf{W} = \mathsf{W}^{(0)} + \mathsf{W}^{(1)} + \mathsf{W}^{(2)}$ [18]. The tensors $\mathsf{W}^{(n)}$ $(n = 0, 1, 2)$ transform under rotations according to the $(2n+1)$-dimensional irreducible representations $\mathsf{D}^{(n)}$ of the rotation group: this means that the new components of a particular $\mathsf{W}^{(n)}$ are linear combinations of the old components of the same $\mathsf{W}^{(n)}$ only.

The decomposition of T into irreducible parts is

$$\begin{aligned} \mathsf{T}^{(0)} &= \frac{1}{2}\rho_{\mathrm{e}}\left(\frac{p^2}{z^2}-\frac{1}{3}\mathbf{v}^2\right)\mathsf{e}, \\ \mathsf{T}^{(1)} &= \mathbf{0}, \\ \mathsf{T}^{(2)} &= \rho_{\mathrm{e}}\left(\mathbf{v}\otimes\mathbf{v}-\frac{1}{3}\mathbf{v}^2\mathsf{e}\right). \end{aligned} \qquad (67)$$

As regards the trace, it is easy to show that, from the relations $\mathrm{Tr}(\mathsf{e}) = 3$ and $\mathrm{Tr}(\mathbf{v}\otimes\mathbf{v}) = \mathbf{v}^2$, it follows $\mathrm{Tr}(\mathsf{T}^{(0)}) = (1/2)\rho_{\mathrm{e}}\left(3p^2/z^2-\mathbf{v}^2\right)$ and $\mathrm{Tr}(\mathsf{T}^{(2)}) = 0$.

The radiation pressure of $\mathsf{T}^{(0)}$ on the surface element having normal $\mathbf{n}$ is

$$\mathbf{s}^{(0)} = -\mathsf{T}^{(0)}\cdot\mathbf{n} = \frac{1}{2}\rho_{\mathrm{e}}\left(\frac{1}{3}\mathbf{v}^2-\frac{p^2}{z^2}\right)\mathbf{n}. \qquad (68)$$

This part of $\mathbf{s}$ is called *isotropic*, since it has the direction of $\mathbf{n}$ and its modulus does not depend on $\mathbf{n}$. From the expression of equation (68) it may be observed that the isotropic radiation pressure acts as a compression if $z^2\mathbf{v}^2 < 3p^2$ and as an expansion in the opposite case. For example, in a plane travelling wave the first case holds true, since $z^2\mathbf{v}^2 = p^2$, while in a standing wave there is a compression in a velocity node and an expansion in a pressure node. In any case, from equation (68) it follows

$$\mathbf{s}^{(0)}\cdot\mathbf{n} = \frac{1}{3}\mathcal{K}-\mathcal{U} = \frac{1}{3}\left(2\mathcal{L}-\mathcal{W}\right). \qquad (69)$$

The radiation pressure due to $\mathsf{T}^{(2)}$ is given by

$$\mathbf{s}^{(2)} = -\mathsf{T}^{(2)}\cdot\mathbf{n} = \rho_{\mathrm{e}}\left(\frac{1}{3}\mathbf{v}^2\mathbf{n}-v_{\mathrm{n}}\mathbf{v}\right), \qquad (70)$$

where $v_{\mathrm{n}} = \mathbf{v}\cdot\mathbf{n}$ is the component of $\mathbf{v}$ in the direction of $\mathbf{n}$. It should be remarked that $\mathbf{s}^{(0)}$ is always normal to the surface, while $\mathbf{s}^{(2)}$ contains a part in the direction of $\mathbf{n}$ and a part in the direction of $\mathbf{v}$. The normal component of $\mathbf{s}^{(2)}$ is

$$\mathbf{s}^{(2)}\cdot\mathbf{n} = \rho_{\mathrm{e}}\left(\frac{1}{3}\mathbf{v}^2-v_{\mathrm{n}}^2\right). \qquad (71)$$

Assuming that $\mathbf{v}$ has a constant direction, the quantity of equation (71) vanishes if the orientation of the surface is chosen in such a way that $3v_{\mathrm{n}}^2 = \mathbf{v}^2$, or $3(\cos\theta)^2 = 1$, where θ is the angle between $\mathbf{n}$ and $\mathbf{v}$. In this case equation (71) represents a force per unit area which is tangent to the surface so that $\mathsf{T}^{(2)}$ is the *shear stress* due to acoustic radiation. The right-hand side of equation (71) can be rewritten in the form $-(2/3)\rho_{\mathrm{e}}\mathbf{v}^2P_2(\cos\theta)$, where P_2 is the Legendre polynomial of order 2. Therefore, the mean value of the normal component with respect to directions in half-spaces $0 \le \theta \le \pi/2$ or $\pi/2 \le \theta \le \pi$ vanishes:

$$\int_0^{\pi/2} P_2(\cos\theta)\sin\theta\,\mathrm{d}\theta = \int_{\pi/2}^{\pi} P_2(\cos\theta)\sin\theta\,\mathrm{d}\theta = 0. \qquad (72)$$

The normal component of $\mathbf{s}$ is given by

$$\mathbf{s}\cdot\mathbf{n} =: s_{\mathrm{n}} = -\frac{1}{2}\rho_{\mathrm{e}}\left(\mathbf{v}^2\cos(2\theta)+\frac{p^2}{z^2}\right). \qquad (73)$$

By averaging this expression with respect to directions in either half-space $0 \le \theta \le \pi/2$, or $\pi/2 \le \theta \le \pi$, it may be found the same value as the isotropic part $s_{\mathrm{n}}^{(0)}$:

$$\langle s_{\mathrm{n}}\rangle_\theta = \frac{1}{2}\rho_{\mathrm{e}}\left(\frac{1}{3}\mathbf{v}^2-\frac{p^2}{z^2}\right) = s_{\mathrm{n}}^{(0)}. \qquad (74)$$

The normal component of radiation pressure for particular values of θ is:

$$\begin{aligned} s_{\mathrm{n}} &= -\frac{1}{2}\rho_{\mathrm{e}}\left(\mathbf{v}^2+\frac{p^2}{z^2}\right) = -\mathcal{W} = -\mathcal{K}-\mathcal{U}, \\ &\quad (\theta = 0, \theta = \pi); \\ s_{\mathrm{n}} &= \frac{1}{2}\rho_{\mathrm{e}}\left(\mathbf{v}^2-\frac{p^2}{z^2}\right) = \mathcal{L} = \mathcal{K}-\mathcal{U}, \\ &\quad (\theta = \frac{\pi}{2}); \\ s_{\mathrm{n}} &= -\frac{1}{2}\rho_{\mathrm{e}}\frac{p^2}{z^2} = -\mathcal{U}, \\ &\quad (\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}). \end{aligned} \qquad (75)$$

From equation (73) one finds that for any type of wave the radiation pressure s_{n} is bounded by

$$-\mathcal{W} \le s_{\mathrm{n}} \le \mathcal{L}. \qquad (76)$$

In particular, at the velocity nodes of a standing wave

$$s_{\mathrm{n}} = -\frac{p^2}{2zc} = -\mathcal{U}, \qquad (77)$$

for any surface orientation. For the same wave, at the sound pressure nodes one has

$$s_{\mathrm{n}} = -\frac{1}{2}\rho_{\mathrm{e}}\mathbf{v}^2\cos(2\theta), \quad (-\mathcal{K} \le s_{\mathrm{n}} \le \mathcal{K}). \qquad (78)$$

If at a certain $\mathbf{x}$ the condition $p^2 = z^2\mathbf{v}^2$ (i.e. $\mathcal{L} = 0$) is satisfied, as it happens for instance at any point of a plane progressive wave, the normal radiation pressure is

$$s_{\mathrm{n}} = -\rho_{\mathrm{e}}\mathbf{v}^2(\cos\theta)^2, \quad (-\mathcal{W} \le s_{\mathrm{n}} \le 0). \qquad (79)$$

In the same case, it follows from equation (68) or equation (69) that the isotropic normal radiation pressure is a compression, whose modulus is equal to one third of the energy density: $\mathbf{s}^{(0)}\cdot\mathbf{n} = -\mathcal{W}/3$.

It is evident from the above expressions that the physical meaning of the radiation pressure is fully understood from the point of view of field theory only, i.e. as already remarked, with no reference to a material target whatsoever.

4. Radiation pressure of a plane reflected wave

The radiation pressure will be now computed in a basic physical situation, in order to display its dependence on the particular geometry and boundary condition. Let us consider the radiation pressure produced by a plane, monochromatic, travelling wave, which is partially reflected and partially absorbed by an infinite flat wall Π. Let the surface of the wall be defined by the equation $\mathbf{n}\cdot\mathbf{x}=0$ ($\mathbf{n}$ is the normal unit vector), and the medium be on the negative side of Π: $\mathbf{n}\cdot\mathbf{x}<0$. Expressing the kinetic potential ϕ as a complex quantity for calculation convenience, the boundary condition may be written as

$$Z\mathbf{n}\cdot\nabla\phi(\mathbf{x},t)+\rho_{\mathrm{e}}\frac{\partial\phi(\mathbf{x},t)}{\partial t}=0,\quad \mathbf{x}\in\Pi, \tag{80}$$

where $Z=\zeta\mathrm{e}^{\mathrm{i}\alpha}$ ($\zeta\geq 0,\ \alpha\in\mathsf{R}$) represents the surface specific impedance. The kinetic potential of the wave is written as a superposition of the incident and the reflected wave. Let D be the amplitude of the gas movement of the incident wave and $\mathbf{k}$ be the wave vector, forming an angle ϑ with the normal $\mathbf{n}$ ($\mathbf{k}\cdot\mathbf{n}=\kappa\cos\vartheta$, $\kappa=|\mathbf{k}|$). Writing $\omega=\kappa c$, we have

$$\begin{aligned}&\phi(\mathbf{x},t)=Dc\left[\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}+R\mathrm{e}^{\mathrm{i}(\mathbf{h}\cdot\mathbf{x}-\omega t)}\right],\\&(\mathbf{n}\cdot\mathbf{x}<0),\end{aligned} \tag{81}$$

where $\mathbf{h}=\mathbf{k}-2\kappa\mathbf{n}\cos\vartheta$ is the wave vector of the reflected wave and R is the complex reflection amplitude, expressed in terms of Z by

$$R=\frac{Z\cos\vartheta-z}{Z\cos\vartheta+z}. \tag{82}$$

Writing R in polar form as $R=\xi\mathrm{e}^{\mathrm{i}\beta}$ ($0\leq\xi\leq 1\,,\beta\in\mathsf{R}$), ξ is the ratio between the amplitudes of the two waves, ξ^2 is the reflection coefficient and β is the phase difference on the wall between the reflected wave and the incident one. Besides the normal $\mathbf{n}$, we introduce a second unit vector $\mathbf{t}:=(\mathbf{k}+\mathbf{h})\,/\,|\mathbf{k}+\mathbf{h}|$ orthogonal to $\mathbf{n}$ and laying in the reflecting plane Π. The time-averaged value $S_{\mathrm{n}}=\langle s_{\mathrm{n}}\rangle$ of the normal component of radiation pressure in any position $\mathbf{x}$ can be easily computed from equations (66) and (81):

$$S_{\mathrm{n}}(\mathbf{x})=-\frac{1}{4}\rho_{\mathrm{e}}\omega^2D^2\left(1+\xi^2\right)[1+\cos(2\vartheta)]. \tag{83}$$

The corresponding tangential component $S_{\mathrm{t}}(\mathbf{x})=\langle\mathbf{s}\cdot\mathbf{t}\rangle$ is

$$S_{\mathrm{t}}(\mathbf{x})=\frac{1}{4}\rho_{\mathrm{e}}\omega^2D^2\left(1-\xi^2\right)\sin(2\vartheta). \tag{84}$$

5. Radiation pressure inside an organ pipe

In this Section we shall consider a second case of physical interest, namely that of radiation pressure inside an open organ flue pipe during a steady excitation. Since the technology for a direct measurement of radiation pressure in the audible range of linear acoustics is not yet available, we have performed an *indirect* measurement of the normal component, through the time-averaged values of the squared acoustic pressure and particle velocity, using equation (73). This measurement is only intended as an illustration of the connection between intensimetry and radiation pressure.

5.1. Theoretical basis

The acoustic field inside the organ pipe can be found – for each single frequency component – by assuming that the motion is one-dimensional between the two extremities ($x=0$, $x=l$) and the source is lumped at the mouth end ($x=0$), producing there a pressure excitation $P\mathrm{e}^{-\mathrm{i}\omega t}$ [19]. As regards the boundary conditions, they are usually given in terms of the complex impedances Z_0 and Z_l at the two ends.

All acoustic quantities can be found by solving the following boundary-value problem for the kinetic potential $\phi(x)$, ($0\leq x\leq l$):

$$\left.\begin{aligned}&\Box\phi\,(x,t)=0,\\&\left[Z_0\frac{\partial\phi\,(x,t)}{\partial x}-\rho_{\mathrm{e}}\frac{\partial\phi\,(x,t)}{\partial t}\right]_{x=0}=P\mathrm{e}^{-\mathrm{i}\omega t},\\&\left[Z_l\frac{\partial\phi\,(x,t)}{\partial x}+\rho_{\mathrm{e}}\frac{\partial\phi\,(x,t)}{\partial t}\right]_{x=l}=0.\end{aligned}\right\} \tag{85}$$

The solution can be written in the form

$$\phi\,(x,t)=2A\sinh\left(\mathrm{i}kx+\varphi\right)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{86}$$

where A and φ are complex constants and $\omega=kc$, the excitation circular frequency, is real. Substituting this solution into the boundary conditions, one finds

$$\begin{aligned}A&=\frac{-\mathrm{i}P}{2k(z\sinh\varphi+Z_0\cosh\varphi)},\\Z_l&=z\tanh\left(\varphi+\mathrm{i}kl\right).\end{aligned} \tag{87}$$

Here, the second equation can be solved for φ to give

$$\varphi=\frac{1}{2}\log\frac{z+Z_l}{z-Z_l}-\mathrm{i}kl. \tag{88}$$

Taking into account the relationship connecting the reflection amplitude at the upper end, $R_l=\xi_l\mathrm{e}^{\mathrm{i}\beta_l}$, to the impedance Z_l, i.e.

$$R_l=\frac{Z_l-z}{Z_l+z}, \tag{89}$$

the constant φ can be expressed in terms of R_l as $\varphi=\chi_l+\mathrm{i}\psi_l$, with

$$\chi_l=-\frac{1}{2}\log\xi_l\,,\quad \psi_l=\frac{1}{2}\left(\pi-\beta_l\right)-kl\,. \tag{90}$$

We remark that φ depends only on the boundary condition at $x=l$, while the amplitude A depends on both boundary conditions.

Let us now see how this simple model gives us the possibility of employing the radiation pressure relations to determine experimentally the reflection properties of the upper end. Expressing A in polar form, $A = |A|\,\mathrm{e}^{\mathrm{i}\theta}$, and putting $D = (|A|\,/c)\mathrm{e}^{\chi_l}$, equation (86) can be written as

$$\phi\,(x,t) = Dc\,\{\exp\left[\mathrm{i}(kx - \omega t + \psi_l + \theta)\right] - \exp\left[-\mathrm{i}(kx + \omega t + \psi_l - \theta) - 2\chi_l\right]\}\,, \quad (91)$$

which, redefining the time variable as $t' = t - (\psi_l + \theta)/\omega$ and using the substitution

$$\beta_l' = \pi - 2\psi_l = \beta_l + 2kl\,, \quad (92)$$

takes the more familiar form

$$\phi(x,t) = Dc\,\{\exp\left[\mathrm{i}(kx - \omega t')\right] + \xi_l \exp\left[-\mathrm{i}(kx + \omega t' - \beta_l')\right]\}\,. \quad (93)$$

Thus the solution of problem (85) is simply given by the superposition of
an incident plane wave $\phi_1 = Dc\exp\left[\mathrm{i}(kx - \omega t')\right]$
and a reflected wave $\phi_2 = \xi_l Dc\exp\left[-\mathrm{i}(kx + \omega t' - \beta_l')\right]$;
it is therefore a particular case of the phenomenon already treated in the previous Section. As we are going to see, the complex reflection amplitude $R_l = \xi_l \mathrm{e}^{\mathrm{i}\beta_l}$ of the upper boundary of the pipe can be determined by performing intensimetric measurements of radiation pressure in the field expressed by equation (93). On the other hand, thanks to equation (89), the study of R_l is equivalent to the study of Z_l, so that, we may focus our attention on the former parameter only.

Since $p = -\rho_\mathrm{e}\,\mathrm{Re}\,(\partial\phi/\partial t)$ and $v = \mathrm{Re}\,(\partial\phi/\partial x)$, the time averaged second order quantities $\langle p^2\rangle$, $\langle v^2\rangle$ and $J = \langle pv\rangle$ are obtained from equation (93) as:

$$\langle p^2\rangle = \frac{(z\omega D)^2}{2}\left[1 + \xi_l^2 + 2\xi_l\cos\left(2kx - \beta_l'\right)\right], \quad (94)$$

$$\langle v^2\rangle = \frac{(\omega D)^2}{2}\left[1 + \xi_l^2 - 2\xi_l\cos\left(2kx - \beta_l'\right)\right], \quad (95)$$

$$J = \frac{z(\omega D)^2}{2}\left(1 - \xi_l^2\right). \quad (96)$$

Using equations (94), (95) and equation (75), the normal component of mean radiation pressure on a plane orthogonal to the pipe axis can be easily computed as

$$S_{n\perp} := \langle s_\mathrm{n}\rangle_{\theta=0} = -W = -\frac{\rho_\mathrm{e}}{2}\left(\omega D\right)^2\left(1 + \xi_l^2\right), \quad (97)$$

where $W = \langle\mathcal{W}\rangle$. From this expression and equation (96), we find the incident wave amplitude D and the reflection modulus ξ_l as:

$$D = \frac{1}{\omega}\sqrt{\frac{J - cS_{\mathrm{n}\perp}}{z}}, \quad \xi_l = \sqrt{\frac{-J - cS_{\mathrm{n}\perp}}{J - cS_{\mathrm{n}\perp}}}, \quad (98)$$

which can be determined experimentally by means of the intensimetric technique.

For the determination of the phase shift β_l, the quantity $\langle s_\mathrm{n}\rangle$ has to be measured on a plane oriented in a direction different from the previous one; in particular, it is convenient to choose a plane parallel to the wave axis, so that the normal radiation pressure on it is given by the time average Lagrangian density $L = \langle\mathcal{L}\rangle$ (see equation 75):

$$S_{\mathrm{n}||}(x) := \langle s_\mathrm{n}\rangle_{\theta=\pi/2} = L(x) = -\rho_\mathrm{e}\omega^2 D^2 \xi_l\cos\left(2kx - \beta_l'\right). \quad (99)$$

From equations (99) and (98) it follows

$$\beta_l' = 2kx - \arccos\left(-\frac{cS_{\mathrm{n}||}(x)}{\sqrt{c^2 S_{\mathrm{n}\perp}^2 - J^2}}\right), \quad (100)$$

and thanks to equation (92) one finally finds

$$\beta_l = 2k(x-l) - \arccos\left(-\frac{cS_{\mathrm{n}||}(x)}{\sqrt{c^2 S_{\mathrm{n}\perp}^2 - J^2}}\right). \quad (101)$$

5.2. Experimental setup and measurements

Through the simple model explained above, and in particular by direct use of equations (98) and (101), it is possible to determine the reflection properties of each frequency component of the pipe simply by performing indirect measurements of radiation pressure in a given point inside the pipe. In order to test its effectiveness, the procedure was applied to a wooden pipe of length $l = 1.85$ m, with open extremity, internal square cross section of 0.1×0.1 m^2 and a fundamental frequency of 82.5 Hz (the musical note E in the 8 ft rank). A series of measurements of the inner field were performed following the well-known two-microphones technique developed for intensity measurements [2]: the study was carried out under steady sound conditions, while air-supplying the pipe by means of a small blowing machine. In order to insert the intensity probe (an axial *B&K 4135* in the side-by-side configuration) into the pipe, a row of holes has been made on one of its sides. They were 40 mm distant from one another, with a diameter of about 8 mm. The two probe microphones, tightly fixed together in a plastic holder, were then inserted into pairs of adjacent holes, as shown in Figure 1: the non-standard distance of 40 mm between adjacent holes was set for keeping minimum errors up to about 2 kHz. During each measurement, all the unused holes were carefully stopped by bolts and sealed by rubber rings, in such a way as to prevent air leaks.

The determination of the average acoustic pressure, particle velocity and mean intensity levels was carried out by means of a *B&K 2133* intensity meter: the readout, in third-octave bands, ranged from 46 Hz to 2.74 kHz, and the integration time was kept sufficiently long (30 s), so as to maintain statistical errors below 0.1 dB for all the quantities.

Since measurements of particle velocity and sound intensity (the latter required from equations (98)–(101)) in a low frequency standing wave field are sensitive to the channel phase mismatch, data were collected adopting the probe reversal technique [20]: two values were taken for both quantities, corresponding to the two opposite probe

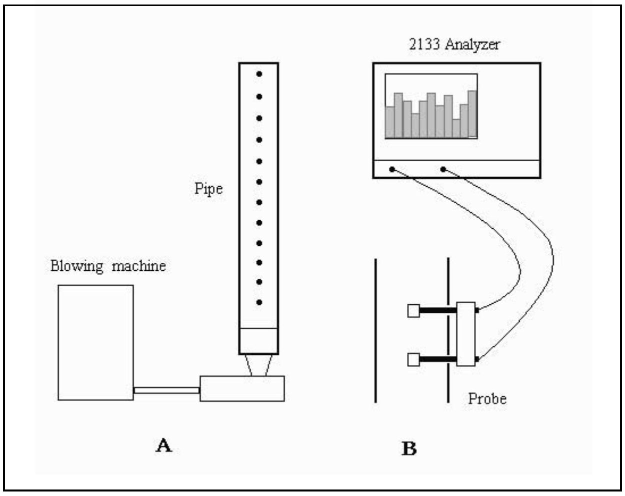

Figure 1. Sketch of the experimental setup used for the measurements in the organ pipe.

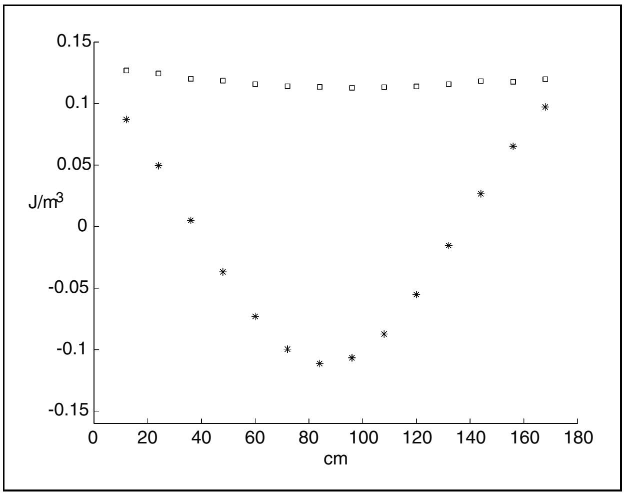

Figure 2. Measures of mean values of radiation pressure along the pipe axis. Asterisks: $S_{n||}$ (Lagrangian density L), squares: $-S_{n\perp}$, (energy density W).

axis directions. Then, the corrected mean squared velocity and intensity were obtained by calculating respectively the half sum and the half difference of the two values.

Figure 2 reports the spatial behaviors of $S_{\mathrm{n}||}$ and $S_{\mathrm{n}\perp}$ obtained by measurements performed on a subset of 14 positions $0.12\,\mathrm{m}$ apart (x ranging from 0.12 to $1.68\,\mathrm{m}$ with respect to the bottom extremity). $S_{\mathrm{n}||}$ shows a sinusoidal dependence on position with just one minimum near the middle, in agreement with equation (99): this means that the level of the fundamental component of the field ($\lambda \simeq 4.2\,\mathrm{m}$) is predominant. On the other hand, the behavior of $S_{\mathrm{n}\perp}$ exhibits a weak but statistically error-free dependence on x. Therefore, the prevision of equation (97), $W(x) = \mathrm{const}$, made on the basis of the simple interference field model (see equation 93), though not perfectly fulfilled, is reasonably good. Anyway, we decided to use the data of the last measurement point ($x = 1.68$ m), that is the one closest to the pipe end, where the sound reflection occurs.

Finally, in order to evaluate the number of frequency components which could be effectively employed for the determination of the reflection parameters, the sound pressure level spectrum at the measurement point was measured by means of an FFT analyzer (Figure 3). As it can be seen, the level of the fundamental frequency ($f_1 = 82.5\,\mathrm{Hz}$) is about 15 dB higher than the levels of the second and third harmonic components ($f_2 = 165\,\mathrm{Hz}$, $f_3 = 247.5\,\mathrm{Hz}$), while the remaining ones are negligible. It is thus reasonable to limit the computation to the first three frequencies only, since these dominate the overall spectrum.

The estimations of ξ_l and β_l for f_1, f_2, f_3 are reported in the last two rows of Table I. These have been obtained from equations (98) and (101) using the values of J, $\langle p^2\rangle$ and $\langle v^2\rangle$ reported in the first three rows of the Table, which correspond to the three third-octave bands centered at 80, 160, 250 Hz respectively. These results are in fairly good agreement with the theoretical values corresponding to a perfect pressure release surface, $\xi_l = 1$, $\beta_l = \pi$, especially for the fundamental frequency.

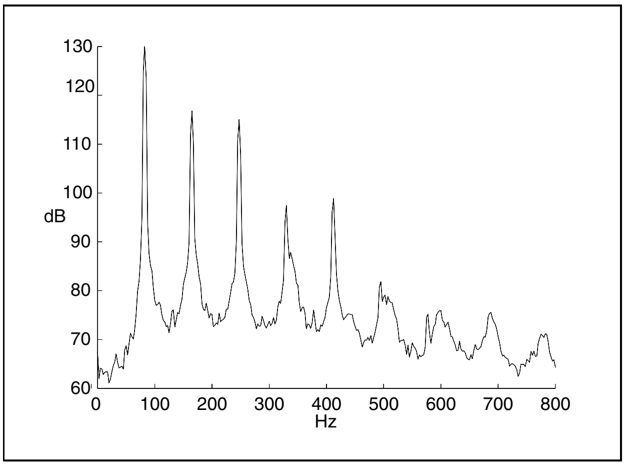

Figure 3. Spectrum of sound pressure level ($\langle p^2\rangle$) of the inner field of the organ pipe.

6. Conclusions

The expression of wave-momentum flux density has been derived in the context of linear field theory from the invariance properties of the Lagrangian density. In particular, using the 4-dimensional formalism of Minkowski spacetime, momentum has been related to the spatial part of the acoustic energy-momentum tensor in the same manner as energy density is linked to the temporal part of the tensor expressed by the energy flux density. In this way, the acoustic radiation pressure is a second order vector quantity which is produced by the momentum flux of the wave field.

Besides its conceptual importance for a correct understanding of the acoustic field from the point of view of the conservation properties in the framework of linear acoustics, it has been shown that radiation pressure is a quantity of practical interest when the interaction of the sound field with its environment can be described by means of a boundary condition involving the surface impedance. In

Table I. The first three rows report respectively the values of J, $S_{n\perp}$ and $S_{n||}$ corresponding to the first three harmonics; at the bottom, the frequency dependence of the reflection modulus ξ_l and phase angle β_l (in degrees) of the steady sound inside the organ pipe is reported. The relative uncertainty of J, $S_{n\perp}$ and $S_{n||}$ is about 1 percent.

f (Hz)	$f_1 = 82.5$	$f_2 = 165$	$f_3 = 247.5$
J(mW/m^2)	41.7	0.577	1.67
$S_{n\perp}$ (mPa)	-120	-37.6	-0.115
$S_{n\|\|}$ (mPa)	97.2	0.114	-0.0332
ξ_l ($\times 10^3$)	999 ± 0.1	995.5 ± 0.2	959 ± 2
$-\beta_l$ (deg)	174 ± 1	166 ± 1	161 ± 1

this context, the relationship between impedance and radiation pressure has been exploited for the basic model describing the reflection of a plane monochromatic wave with standard boundary conditions (equation 80). A similar relationship, as shown in Sect. 5, can be used for determining the complex reflection amplitude at the top of an open organ pipe, from the indirect local measurement of the inner field radiation pressure during a steady excitation.

Appendix

A1. Angular momentum conservation

Another interesting example illustrating the relation between invariance properties of the Lagrangian $\mathcal{L}$ in the field theoretical approach and conservation laws is treated here by considering the acoustic Lorentz transformations

$$\tilde{x}^\mu = A^\mu_\nu x^\nu \,, \quad (\mu, \nu = 0, 1, 2, 3), \tag{A1}$$

where $\{A^\mu_\nu\}$ is a matrix leaving invariant the metric tensor of space M_4. From the invariance of $\mathcal{L}$ with respect to these transformations, a conservation law is obtained, which is again written as a condition of vanishing divergence

$$\partial_\lambda M^{\lambda\mu\nu} = 0, \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{A2}$$

of the tensor $M \in \mathsf{M}_4 \otimes \mathsf{M}_4 \otimes \mathsf{M}_4$ with components

$$M^{\lambda\mu\nu} := T^{\lambda\mu} x^\nu - T^{\lambda\nu} x^\mu.$$

It is easily seen that for condition of equation (A2) to hold, it is necessary and sufficient that the tensor T be symmetric: $T^{\lambda\mu} = T^{\mu\lambda}$. In particular, the rotations of E_3 form a subgroup of the Lorentz group and the corresponding conserved quantity can be defined by means of the *acoustic angular momentum density*

$$\mathbf{l} = \frac{1}{2}\varepsilon_{hij}\left(x^i q^j - x^j q^i\right)\mathbf{e}^h, \tag{A3}$$

where q^i are the components of the vector of equation (48) and ε_{hij} are the components of the fully antisymmetric Ricci tensor defined by

$$\varepsilon_{hij} := \frac{\mathbf{e}_h \cdot (\mathbf{e}_i \wedge \mathbf{e}_j)}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3)} \tag{A4}$$

$$\begin{cases} 1, & (h,i,j) \text{even permutation of } (1,2,3), \\ -1, & (h,i,j) \text{odd permutation of } (1,2,3), \\ 0, & (h,i,j) \text{not a permutation of } (1,2,3). \end{cases}$$

To find the vector $\mathbf{l}$ from the tensor T, we write the components $\lambda = 0,\ \mu = i = 1,2,3,\ \nu = j = 1,2,3$ of M and get

$$\begin{aligned} M^{0ij} &= T^{0i} x^j - T^{0j} x^i = -c\left(x^i q^j - x^j q^i\right) \\ &= -cL^{ij}, \qquad (i,j = 1,2,3), \end{aligned} \tag{A5}$$

where the antisymmetric tensor L^{ij} is related to $\mathbf{l}$ by

$$l_h = \frac{1}{2}\varepsilon_{hij} L^{ij}\,. \tag{A6}$$

The angular momentum conservation law is found by choosing in equation (A2) the components $\mu = i = 1,2,3$, $\nu = j = 1,2,3$:

$$\begin{aligned} \frac{1}{c}\frac{\partial M^{0ij}}{\partial t} &= -\left(\frac{\partial M^{1ij}}{\partial x^1} + \frac{\partial M^{2ij}}{\partial x^2} + \frac{\partial M^{3ij}}{\partial x^3}\right) \\ &= -\partial_k M^{kij}, \end{aligned} \tag{A7}$$

which can be rewritten as

$$\frac{\partial L^{ij}}{\partial t} = -\partial_k M^{kij}. \tag{A8}$$

Here, the left-hand side is related to the time derivative of the angular momentum density. In integral form, the conservation law has either form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{l}(\mathbf{x},t)\mathrm{d}^3\mathbf{x} = \\ &-\frac{1}{2}\varepsilon_{hij}\mathbf{e}^h \int_V \partial_k M^{kij}(\mathbf{x},t)\mathrm{d}^3\mathbf{x}\,, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathbf{l}(\mathbf{x},t)\mathrm{d}^3\mathbf{x} = \\ &-\frac{1}{2}\varepsilon_{hij}\mathbf{e}^h \int_S n_k M^{kij}(\mathbf{x},t)\mathrm{d}^2\mathbf{x}, \end{aligned} \tag{A9}$$

where the two quantities

$$-\frac{1}{2}\left(\varepsilon_{hij}\partial_k M^{kij}\mathbf{e}^h\right), \quad -\frac{1}{2}\left(\varepsilon_{hij} n_k M^{kij}\mathbf{e}^h\right), \tag{A10}$$

respectively represent the *volume density* and the *surface density of the moment of a force*. The right-hand side of equation (A9) is interpreted as the torque acting on the surface of a body hit by acoustic radiation; for its correct computation, the boundary conditions on the surface should be taken into account. An example of an acoustic field with angular momentum has been given by Schroeder, who also demonstrated experimentally the existence of a torque acting on an absorbing body set into rotation by the radiation [21].

A2. Comparison with electromagnetic radiation pressure

The electromagnetic radiation pressure is obtained from the Maxwell stress tensor [6], [22] through the same general principles as the acoustic radiation pressure. Nevertheless, the comparison between the two quantities is not trivial since some properties are specific of the two cases.

From the Lagrangian density of the electromagnetic field in vacuum (Gauss units)

$$\mathcal{L} = \frac{1}{8\pi}\left(\mathbf{E}^2 - \mathbf{H}^2\right), \tag{A11}$$

written in terms of electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$, one finds the energy-momentum tensor

$$\{T^{\mu\nu}\} = \begin{pmatrix} \mathcal{W} & \mathbf{t} \\ \mathbf{t} & \mathsf{T} \end{pmatrix}; \tag{A12}$$

here $\mathcal{W}$ is the electromagnetic energy density, $\mathbf{t} = \mathbf{j}/c_l$ is related to the Poynting vector $\mathbf{j}$, c_l is the speed of light and T the electromagnetic momentum flux density:

$$\begin{aligned} \mathcal{W} &= \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{H}^2\right), \\ \mathbf{j} &= \frac{c_l}{4\pi}\mathbf{E}\wedge\mathbf{H}, \\ \mathsf{T} &= \mathcal{W}\mathsf{e} - \frac{1}{4\pi}\left[\mathbf{E}\otimes\mathbf{E} + \mathbf{H}\otimes\mathbf{H}\right]. \end{aligned} \tag{A13}$$

The decomposition of T into irreducible parts is the following

$$\begin{aligned} \mathsf{T}^{(0)} &= \frac{1}{3}\mathcal{W}\mathsf{e}, \qquad \mathsf{T}^{(1)} = \mathbf{0}, \\ \mathsf{T}^{(2)} &= \frac{2}{3}\mathcal{W}\mathsf{e} - \frac{1}{4\pi}\left[\mathbf{E}\otimes\mathbf{E} + \mathbf{H}\otimes\mathbf{H}\right]. \end{aligned} \tag{A14}$$

The radiation pressure on a surface with normal $\mathbf{n}$ is now

$$\mathbf{s} = -\mathsf{T}\cdot\mathbf{n} = -\mathcal{W}\mathbf{n} + \frac{1}{4\pi}\left(E_\mathrm{n}\mathbf{E} + H_\mathrm{n}\mathbf{H}\right), \tag{A15}$$

where the subscript n denotes the normal component. The isotropic and shear parts are

$$\begin{aligned} \mathbf{s}^{(0)} &= -\frac{1}{3}\mathcal{W}\mathbf{n}, \\ \mathbf{s}^{(2)} &= -\frac{2}{3}\mathcal{W}\mathbf{n} + \frac{1}{4\pi}\left(E_\mathrm{n}\mathbf{E} + H_\mathrm{n}\mathbf{H}\right). \end{aligned} \tag{A16}$$

The double inequality of equation (76) is now replaced by

$$-\mathcal{W} \le \mathbf{s}\cdot\mathbf{n} \le \mathcal{W}, \tag{A17}$$

where the first becomes an equality when $E_\mathrm{n} = H_\mathrm{n} = 0$ (normal incidence of a plane wave), the second when $E_\mathrm{n}^2 = \mathbf{E}^2$, $H_\mathrm{n}^2 = \mathbf{H}^2$ ($\mathbf{E}$ and $\mathbf{H}$ parallel to $\mathbf{n}$, as in the case of a purely electrostatic field on a metal surface).

It is remarkable that $\mathbf{s}^{(0)}\cdot\mathbf{n}$ is always a compression, equal to one third of the energy density, while in the acoustic case this is true only in positions where $\mathcal{L} = 0$, i.e. where kinetic and potential energy densities are equal.

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